SIGNALS AND SYSTEMS

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Signal Analysis

Signal and Vectors

- Any vector A in 3 dimensional space can be expressed as

$$A = A_1 a + A_2 b + A_3 c$$

- a, b, c are vectors that do not lie in the same plane and are not collinear

- A_1, A_2, and A_3 are linearly independent

- No one of the vectors can be expressed as a linear combination of the other 2

- a, b, c is said to form a basis for a 3 dimensional vector space

- To represent a time signal or function X(t) on a T interval (t_0 to t_0+T) consider a set of time function independent of x(t) $\cdot_1(t), \cdot_2(t), \cdot_3(t)$ $\cdot_N(t)$

Signal and Vectors

- X(t) can expanded as

$$x_a(t) = \sum_{n=0}^{N} x_n \phi_n(t)$$

- N coefficients X_n are independent of time and subscript x_a is an approximation

Signals and Vectors

- Signal **g** can be written as N dimensional vector

$$\mathbf{g} = [g(t_1)\ g(t_2)\ \ldots\ldots\ldots\ldots\ g(t_N)]$$

- Continuous time signals are straightforward generalization of finite dimension vectors

$$\lim_{N \to \infty} g \to g(t)$$

$$t \in [a, b]$$

- In vector (dot or scalar), inner product of two real-valued vector g and x:
  - $\langle g,x \rangle = ||g||.||x||\cos\theta$ θ - angle between vector **g** and **x**
  - Length of a vector x:

$$||x||_2 = \langle x.x \rangle$$

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Analogy between Signal Spaces and Vector Spaces

- Consider two vectors V1 and V2 as shown in Fig. If V1 is to be
- represented in terms of V2

$$V_1 = C_{12}V_2 + V_e$$

- where Ve is the error.

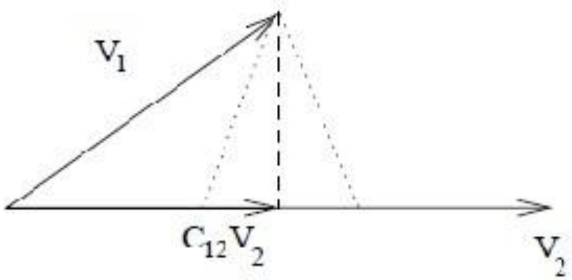

Figure : Representation in vector space

Component of a Vector in terms of another vector.

- Vector **g** in Figure 1 can be expressed in terms of vector **x**

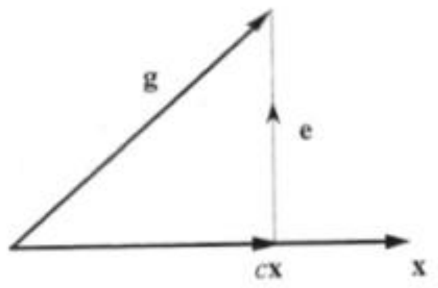

Figure 1

$\mathbf{g} = c\mathbf{x} + \mathbf{e}$

$\mathbf{g} \cdot\ c\mathbf{x}$

$\mathbf{e} = \mathbf{g} - c\mathbf{x}$ (error vector)

- Figure 2 shows infinite possibilities to express vector **g** in terms of vector **x**

Figure 2

$$\mathbf{g} = c_1\mathbf{x} + e_1 = c_2\,\mathbf{x} + e_2$$

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- Let f1(t) and f2(t) be two real signals. Approximation of f1(t) by f2(t) over a time interval t1 < t < t2 can be given by

$$f_e(t) = f_1(t) - C_{12} f_2(t)$$

  where fe(t) is the error function.

- The goal is to find C12 such that fe(t) is minimum over the interval considered. The energy of the error signal ε given by

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)]^2 \, dt$$

  To find C12,

$$\frac{\partial \varepsilon}{\partial C_{12}} = 0$$

- Solving the above equation we get

$$C_{12} = \frac{\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f_1(t).f_2(t)\,dt}{\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f_2^2(t)\,dt}$$

- The denominator is the energy of the signal f2(t).
- When f1(t) and f2(t) are orthogonal to each other
  C12 = 0.

Scalar or Dot Product of Two Vectors

$$\mathbf{g} \cdot \mathbf{x} = |\mathbf{g}||\mathbf{x}| \cos \theta$$

- · is the angle between vectors **g** and **x**.
- The length of the component **g** along **x** is: $c|\mathbf{x}| = |\mathbf{g}| \cos \theta$
- Multiplying both sides by |**x**| yields: $c|\mathbf{x}|^2 = |\mathbf{g}||\mathbf{x}| \cos \theta = \mathbf{g} \cdot \mathbf{x}$
- Where: $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$
- Therefore: $c = \frac{\mathbf{g} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{1}{|\mathbf{x}|^2} \mathbf{g} \cdot \mathbf{x}$
- If g and x are **Orthogonal** (perpendicular): $\mathbf{g} \cdot \mathbf{x} = 0$
- Vectors g and x are defined to be **Orthogonal** if the dot product of the two vectors are zero.

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Components and Orthogonality of Signals

- Concepts of vector component and orthogonality can be extended to CTS
- If signal g(t) is approximated by another signal x(t) as :

$$g(t) \simeq cx(t) \qquad t_1 \leq t \leq t_2$$

- The optimum value of c that minimizes the energy of the error signal is:

$$c = \frac{1}{E_x} \int_{t_1}^{t_2} g(t)x(t)\,dt$$

- We define **real** signals g(t) and x(t) to be orthogonal over the interval $[t_1, t_2]$, if:

$$\int_{t_1}^{t_2} g(t)x(t)\,dt = 0$$

- We define **complex** signals* $x_1(t)$ and $x_2(t)$ to be orthogonal over the interval $[t_1, t_2]$:

$$\int_{t_1}^{t_2} x_1(t)x_2^*(t)\,dt = 0 \qquad \text{or} \qquad \int_{t_1}^{t_2} x_1^*(t)x_2(t)\,dt = 0$$

Example

- For the square signal g(t) find the component in g(t) of the form sin t. In order words, approximate g(t) in terms of sin t so that the energy of the error signal is minimum

$$g(t) \simeq c \sin t \qquad 0 \le t \le 2\pi$$

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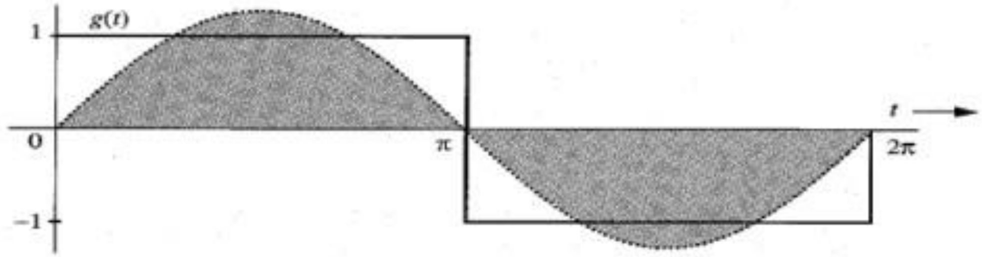


Figure 2.17 Approximation of a square signal in terms of a single sinusoid.

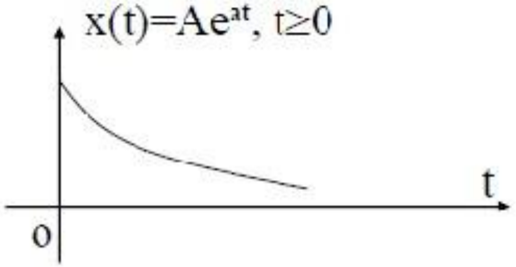
$$x(t) = \sin t \qquad \text{and} \qquad E_x = \int_0^{2\pi} \sin^2 t \, dt = \pi$$

$$c = \frac{1}{\pi} \int_0^{2\pi} g(t) \sin t \, dt = \frac{1}{\pi} \left[\int_0^{\pi} \sin t \, dt + \int_{\pi}^{2\pi} -\sin t \, dt \right] = \frac{4}{\pi}$$

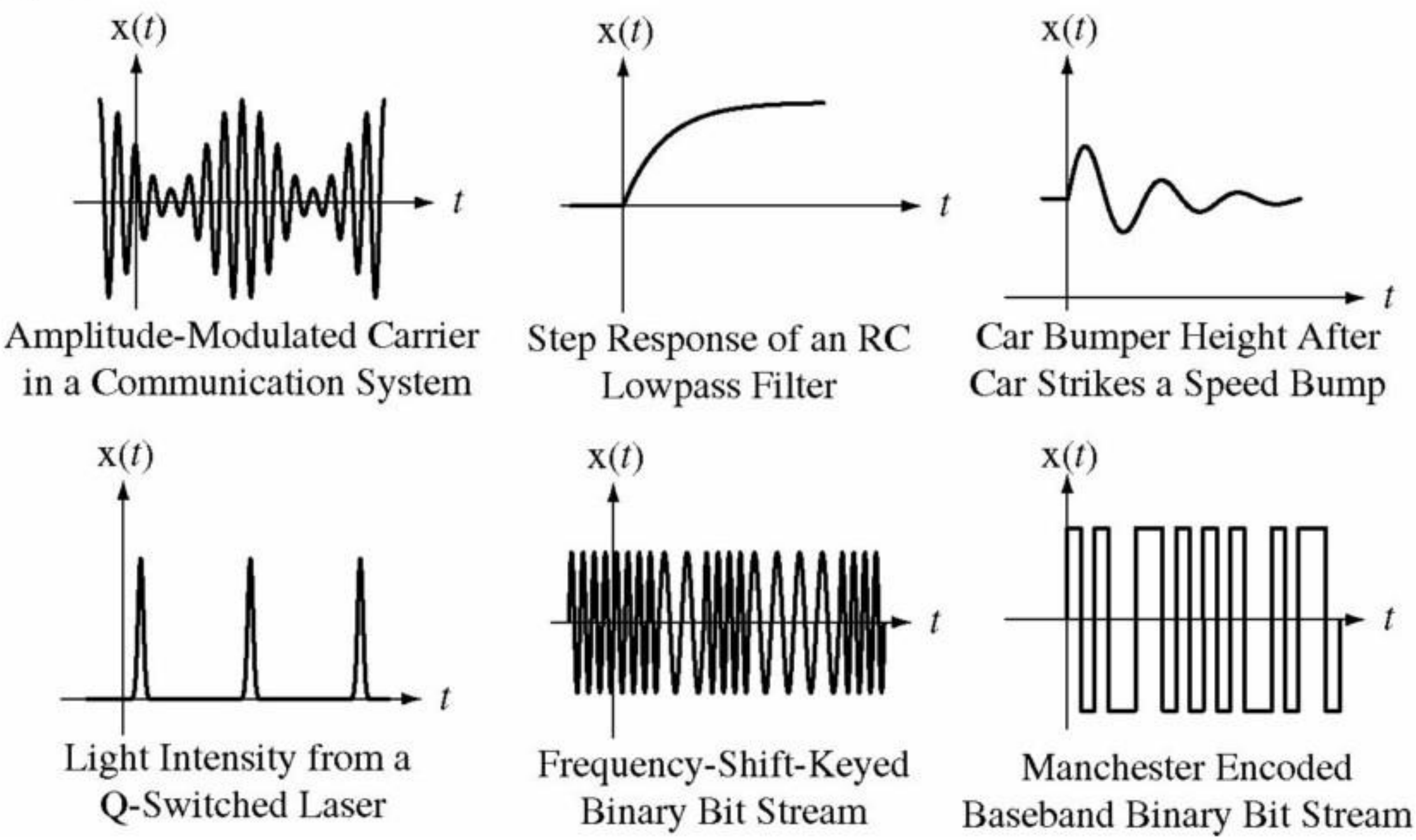
$$g(t) \simeq \frac{4}{\pi} \sin t$$

Introduction to Signals

- A Signal is the function of one or more independent variables that carries some information to represent a physical phenomenon.
- A continuous-time signal, also called an analog signal, is defined along a continuum of time.

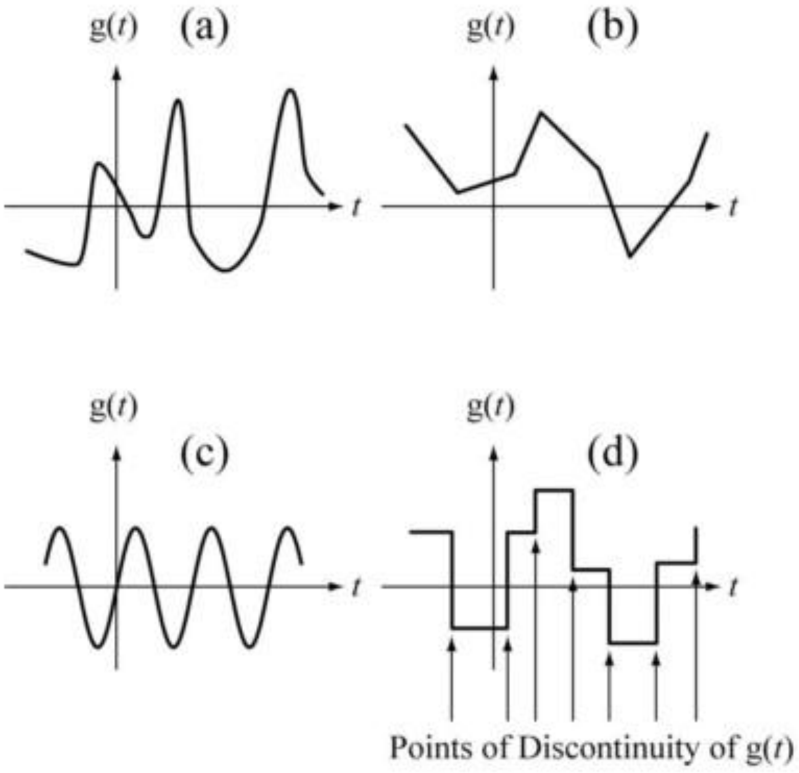


Typical Continuous-Time Signals



Continuous vs Continuous-Time Signals

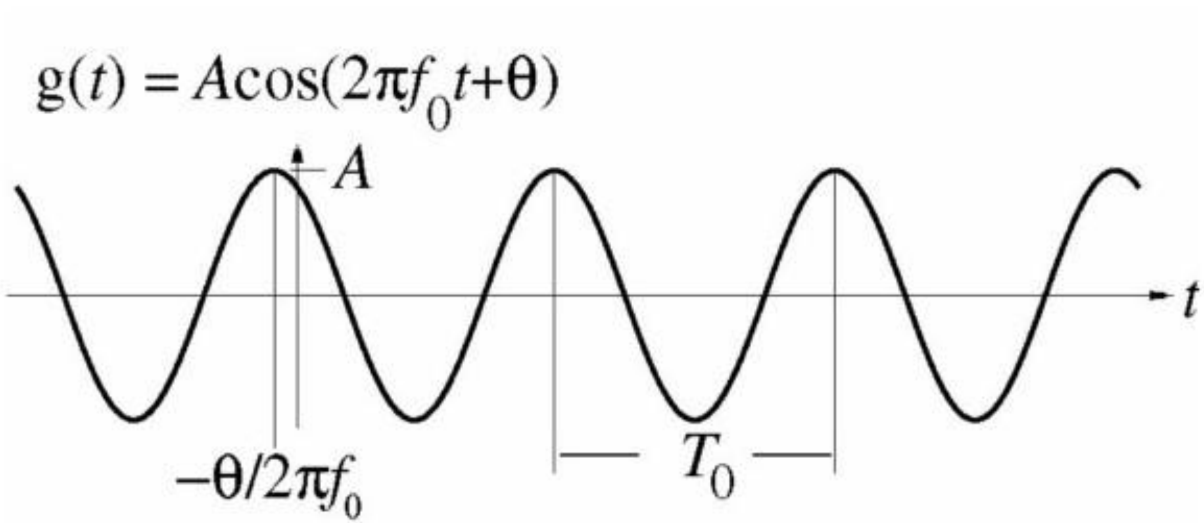
All continuous signals that are functions of time are **continuous-time** but not all continuous-time signals are continuous



Continuous-Time Sinusoids

$$g(t) = A\cos(2\pi t/T_0 + \theta) = A\cos(2\pi f_0 t + \theta) = A\cos(\omega_0 t + \theta)$$

Amplitude	Period (s)	Phase Shift (radians)	Cyclic Frequency (Hz)	Radian Frequency (radians/s)



Elementary Signals

Sinusoidal & Exponential Signals

- Sinusoids and exponentials are important in signal and system analysis because they arise naturally in the solutions of the differential equations.
- Sinusoidal Signals can expressed in either of two ways :

  cyclic frequency *form*- $A \sin 2\Pi f_o t = A \sin(2\Pi/T_o)t$

  radian frequency form- $A \sin \omega_0 t$

  $\omega_0 = 2\Pi f_o = 2\Pi/T_o$

  T_o = Time Period of the Sinusoidal Wave

Sinusoidal & Exponential Signals Contd.

$x(t) = A \sin(2\Pi f_o t + \theta)$
$= A \sin(\omega_o t + \theta)$ } Sinusoidal signal

$x(t) = Ae^{at}$ Real Exponential

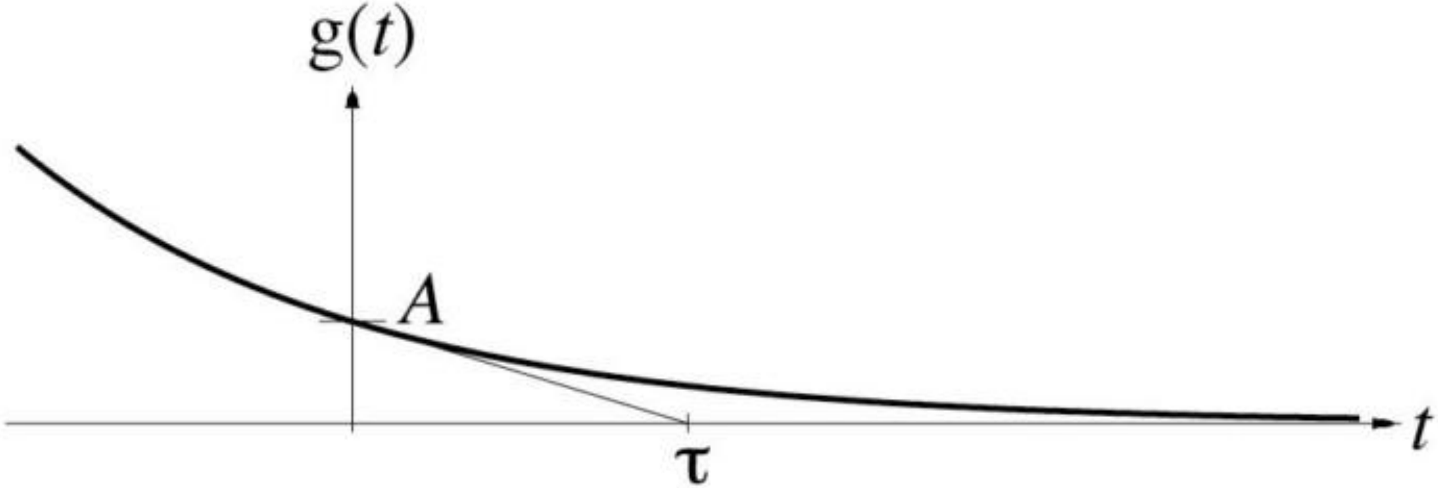
$= Ae^{j\omega t} = A[\cos(\omega_o t) + j \sin(\omega_o t)]$ Complex Exponential

θ = Phase of sinusoidal wave
A = amplitude of a sinusoidal or exponential signal
f_o = fundamental cyclic frequency of sinusoidal signal
ω_o = radian frequency

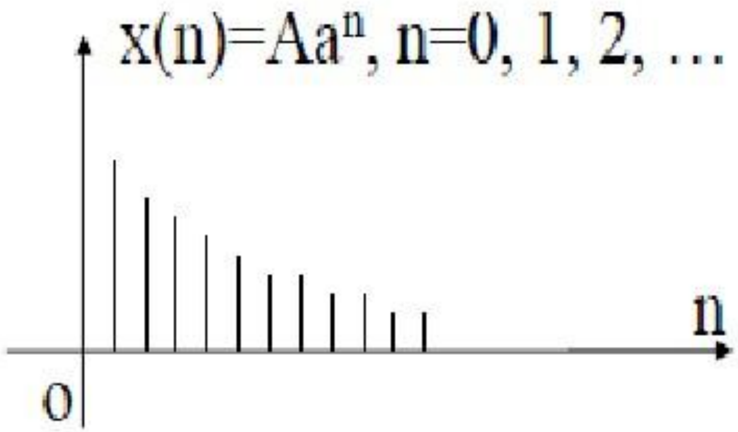
Continuous-Time Exponentials

$$g(t) = Ae^{-t/\tau}$$

Amplitude Time Constant (s)



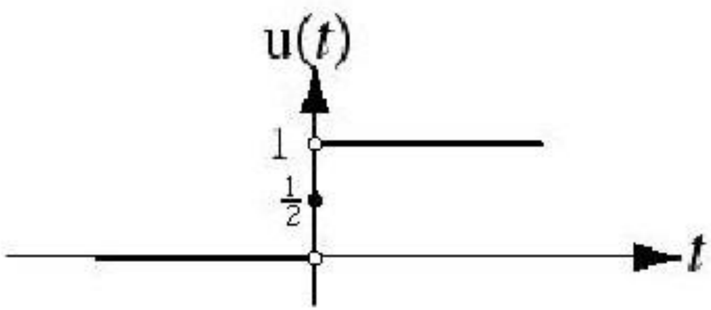
A discrete-time signal is defined at discrete times.



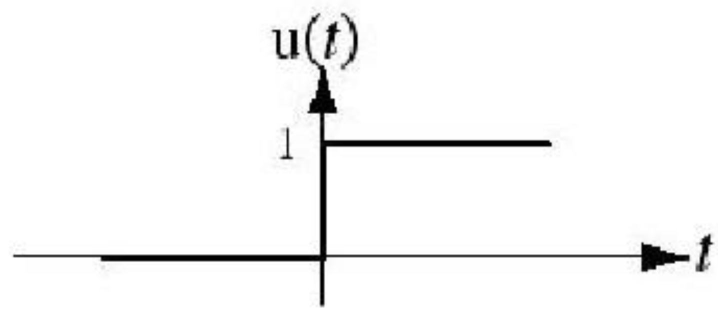
Unit Step Function

$$u(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases}$$

Precise Graph



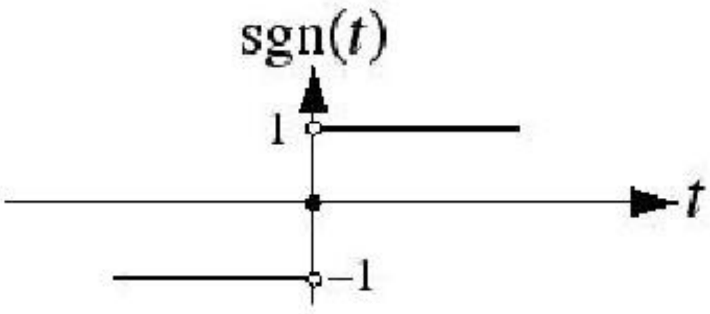
Commonly-Used Graph



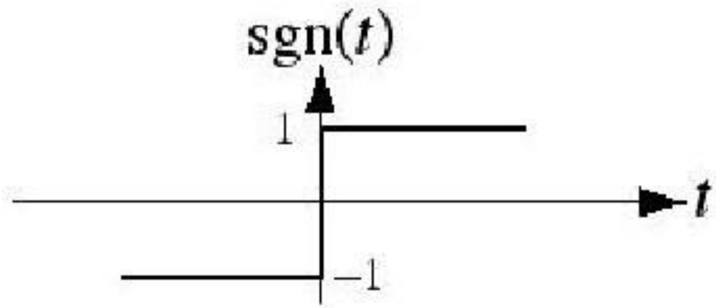
Signum Function

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} = 2u(t) - 1$$

Precise Graph

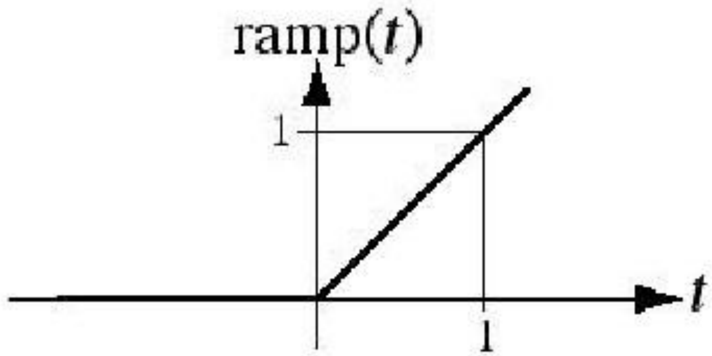


Commonly-Used Graph



The signum function, is closely related to the unit-step function.

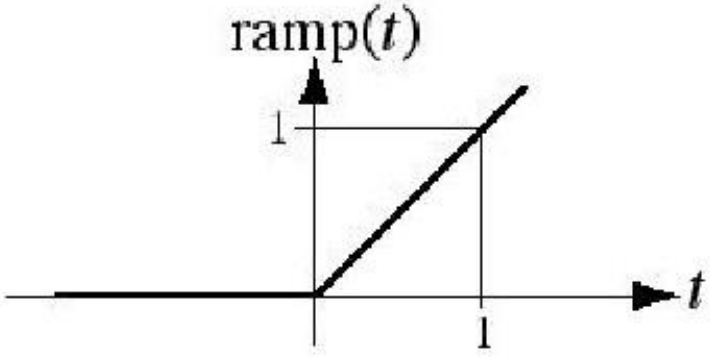
Unit Ramp Function



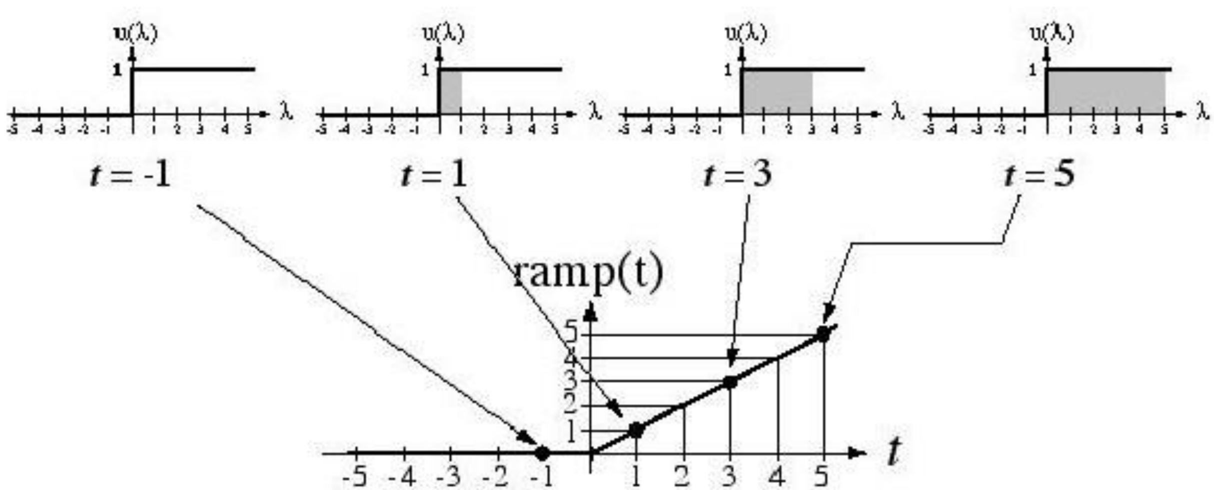
$$\operatorname{ramp}(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} = \int_{-\infty}^{t} u(\lambda)\, d\lambda = t\, u(t)$$

- The unit ramp function is the integral of the unit step function.
- It is called the unit ramp function because for positive t, its slope is one amplitude unit per time.

The Unit Ramp Function



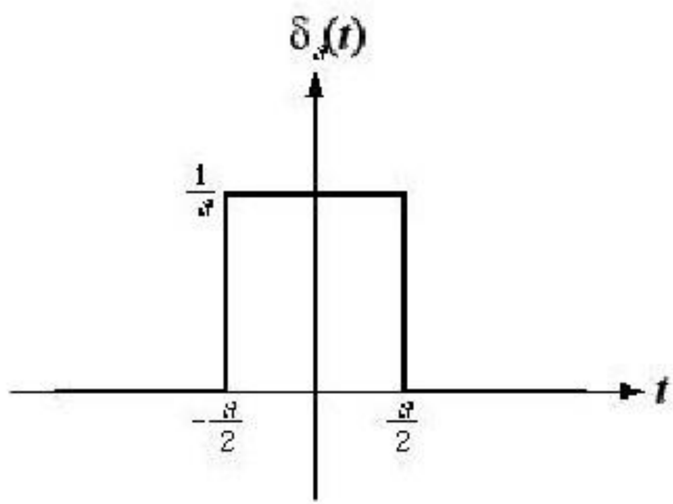
$$\text{ramp}(t) = \begin{cases} t, & t > 0 \\ 0, & t \le 0 \end{cases} = \int_{-\infty}^{t} u(\lambda)\,d\lambda = t\,u(t)$$



Rectangular Pulse or Gate Function

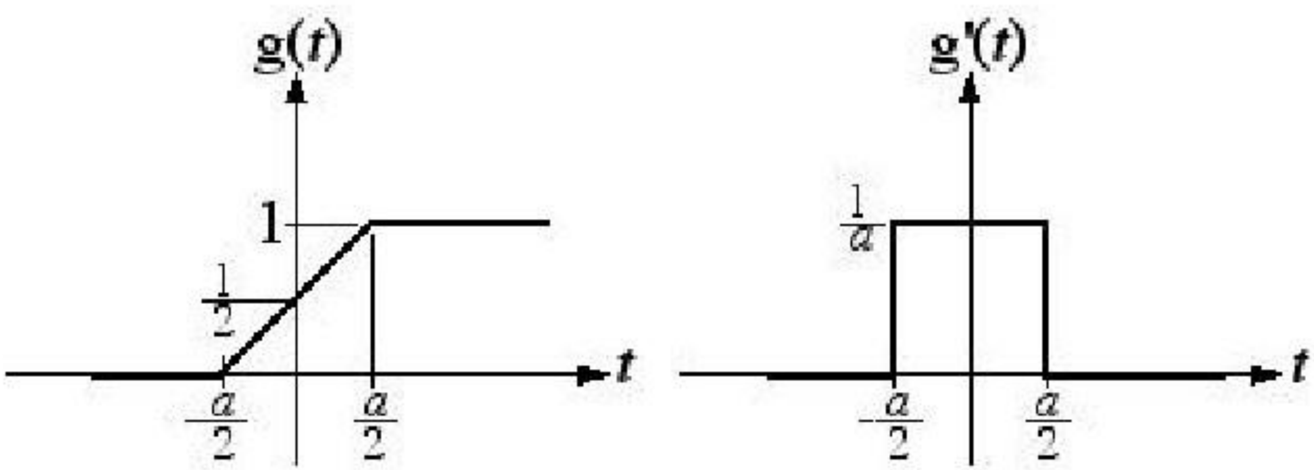
Rectangular pulse,

$$\delta_a(t) = \begin{cases} 1/a, & |t| < a/2 \\ 0, & |t| > a/2 \end{cases}$$



Unit Impulse Function

As a approaches zero, g• t• approaches a unit step and g• approaches a unit impulse• t•

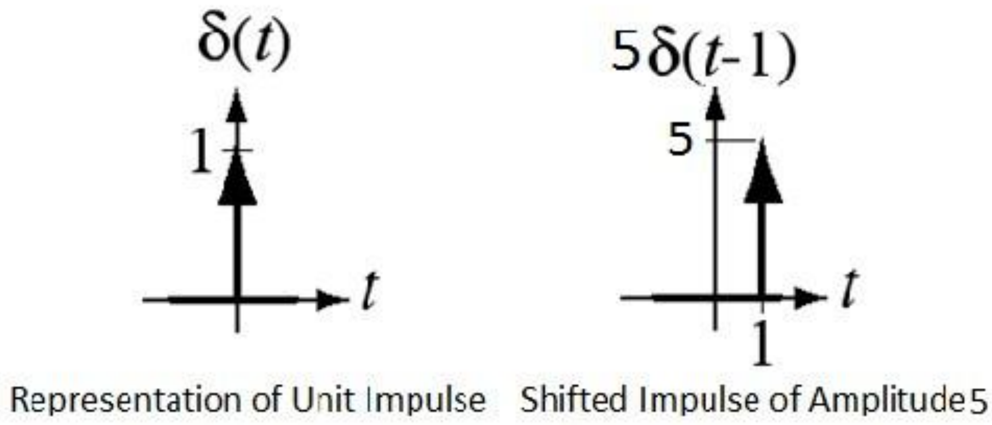


Functions that approach unit step and unit impulse

So unit impulse function is the **derivative** of the unit step function or unit step is the integral of the unit impulse function

Representation of Impulse Function

The area under an impulse is called its strength or weight. It is represented graphically by a vertical arrow. An impulse with a strength of one is called a unit impulse.



Properties of the Impulse Function

The Sampling Property

$$\int_{-\infty}^{\infty} g(t)\,\delta(t - t_0)\,dt = g(t_0)$$

The Scaling Property

$$\delta\big(a(t - t_0)\big) = \frac{1}{|a|}\,\delta(t - t_0)$$

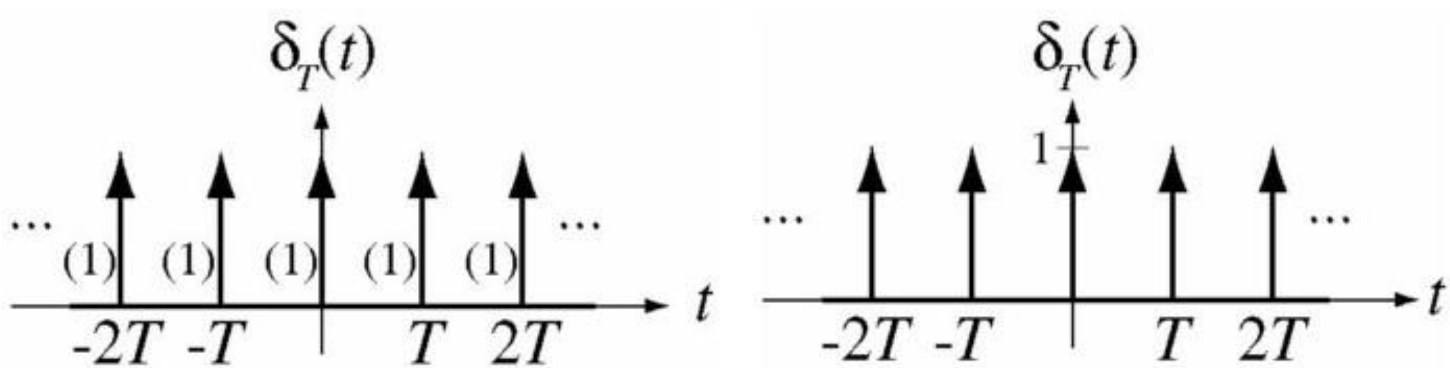
The Replication Property

$$g(t) \otimes \delta(t) = g(t)$$

Unit Impulse Train

The unit impulse train is a sum of infinitely uniformly-spaced impulses and is given by

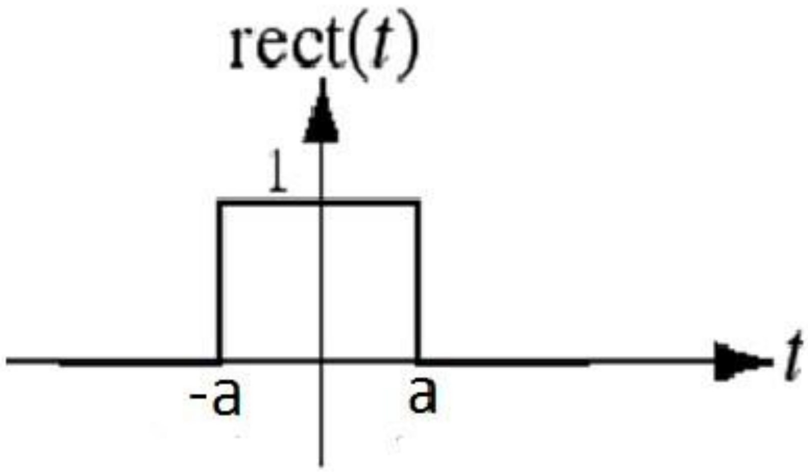
$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad n \text{ an integer}$$



The Unit Rectangle Function

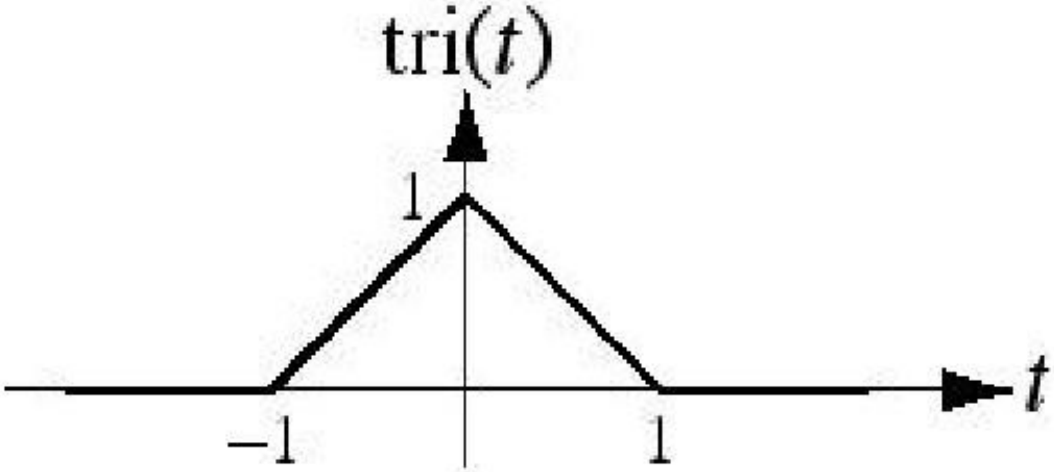
The unit rectangle or gate signal can be represented as combination of two shifted unit step signals as shown

$$\mathrm{rect}(t) = u(t+a) - u(t-a)$$



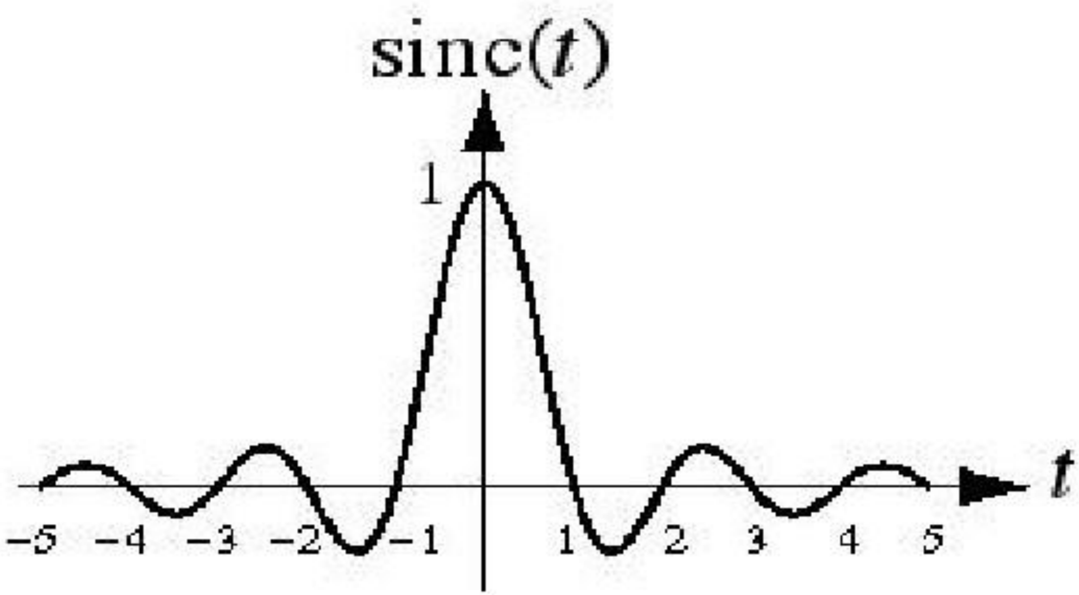
The Unit Triangle Function

A triangular pulse whose height and area are both one but its base width is not, is called unit triangle function. The unit triangle is related to the unit rectangle through an operation called **convolution**.



Sinc Function

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



Discrete-Time Signals

- **Sampling** is the acquisition of the values of a continuous-time signal at discrete points in time
- x(*t*) is a continuous-time signal, x[*n*] is a discrete-time signal

$x[n] = x(nT_s)$ where T_s is the time between samples

Discrete Time Exponential and Sinusoidal Signals

- DT signals can be defined in a manner analogous to their continuous-time counter part

$x[n] = A \sin(2\Pi n/N_o + \theta)$ Discrete Time Sinusoidal Signal

$= A \sin(2\Pi F_o n + \theta)$

$x[n] = a^n$ Discrete Time Exponential Signal

n = the discrete time

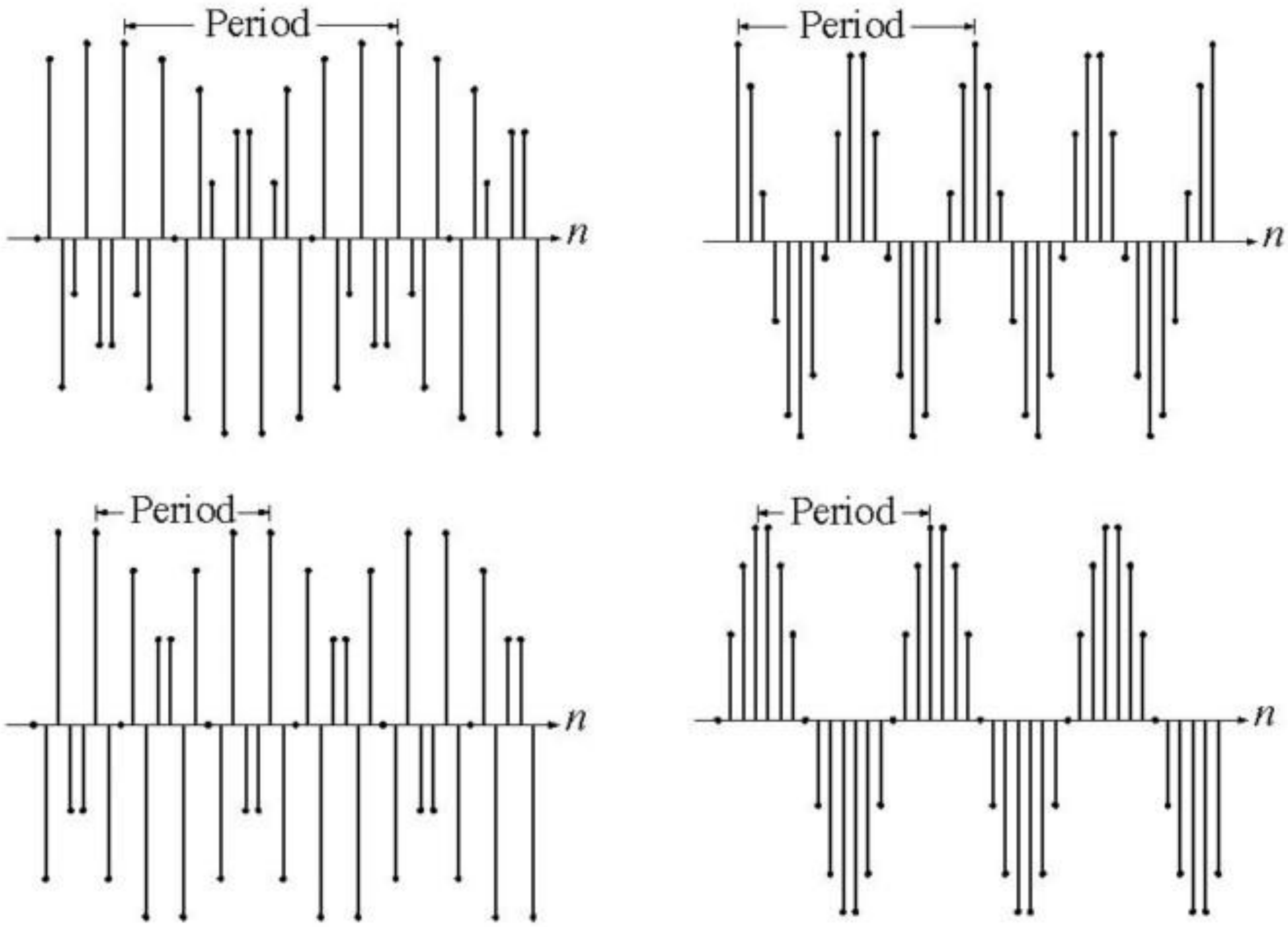
A = amplitude

θ = phase shifting radians,

N_o = Discrete Period of the wave

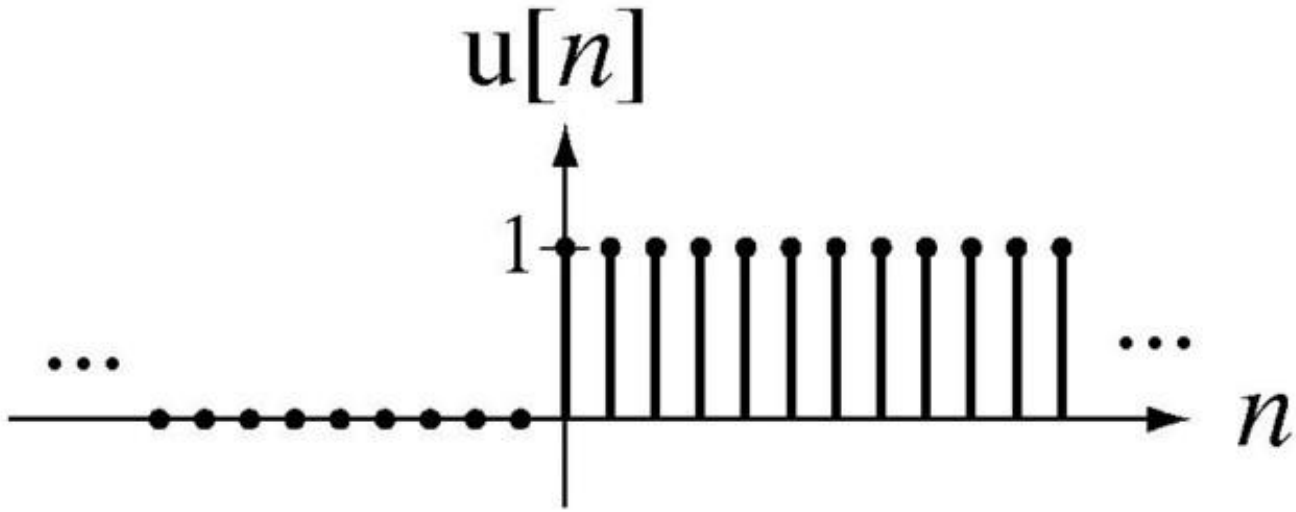
$1/N_0 = F_o = \Omega_o/2\Pi$ = Discrete Frequency

Discrete Time Sinusoidal Signals



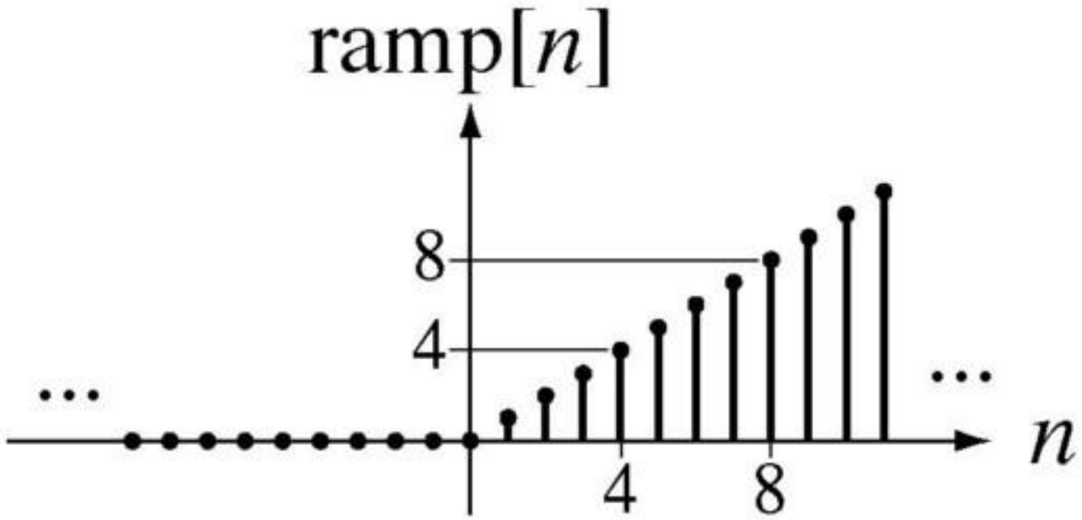
Discrete Time Unit Step Function or Unit Sequence Function

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



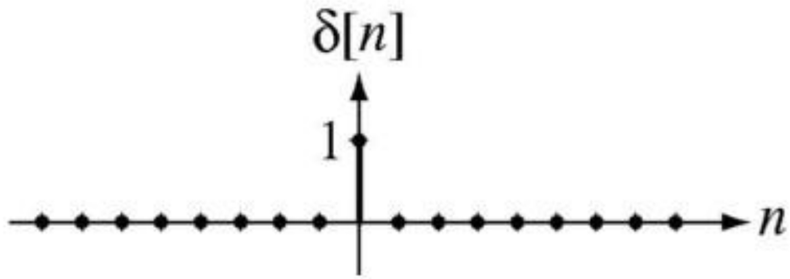
Discrete Time Unit Ramp Function

$$\mathrm{ramp}[n] = \begin{cases} n, & n \ge 0 \\ 0, & n < 0 \end{cases} = \sum_{m=-\infty}^{n} \mathrm{u}[m-1]$$



Discrete Time Unit Impulse Function or Unit Pulse Sequence

$$\cdot\ \cdot\ n \cdot\ \cdot \begin{cases} 1, & n \cdot\ 0 \\ 0, & n\ \cdot\ 0 \end{cases}$$



$\cdot\quad n\quad \cdot\ \cdot\quad an \cdot\quad \cdot\qquad \cdot\qquad \cdot$ for any non-zero, finite

Unit Pulse Sequence Contd.

- The discrete-time unit impulse is a function in the ordinary sense in contrast with the continuous-time unit impulse.
- It has a sampling property.
- It has no scaling property i.e.

$\delta[n]=\delta[an]$ for any non-zero finite integer „a‟

Operations of Signals

- Sometime a given mathematical function may completely describe a signal .
- Different operations are required for different purposes of arbitrary signals.
- The operations on signals can be

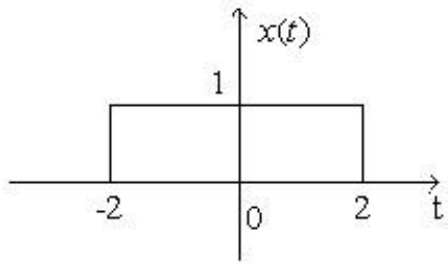
  Time Shifting

  Time Scaling

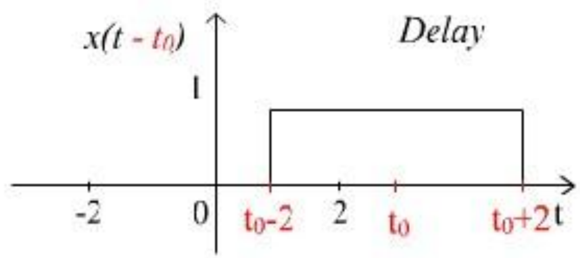
  Time Inversion or Time Folding

Time Shifting

- The original signal $x(t)$ is shifted by an amount t_o.

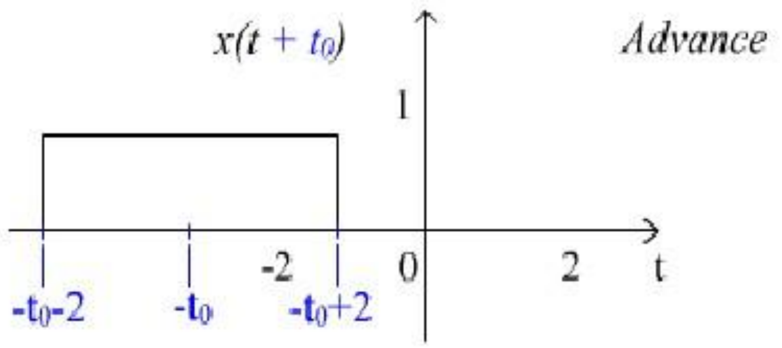


- X(t)· X(t-to) · Signal Delayed· Shift to the right



Time Shifting Contd.

- X(t)· X(t+to) · Signal Advanced· Shift to the left

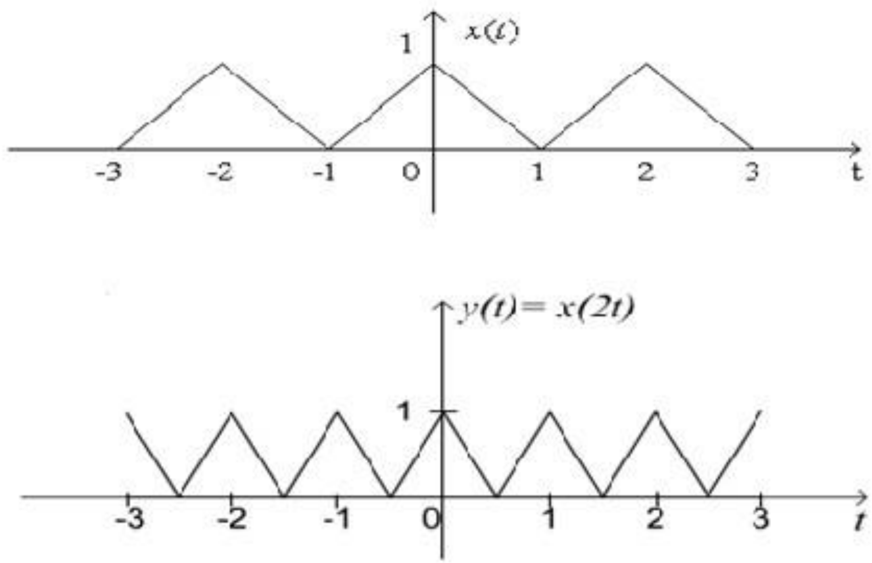


Time Scaling

- For the given function x(t), x(at) is the time scaled version of x(t)
- For a > 1,period of function x(t) reduces and function speeds up. Graph of the function shrinks.
- For a < 1, the period of the x(t) increases and the function slows down. Graph of the function expands.

Time scaling Contd.

Example: Given $x(t)$ and we are to find $y(t) = x(2t)$.

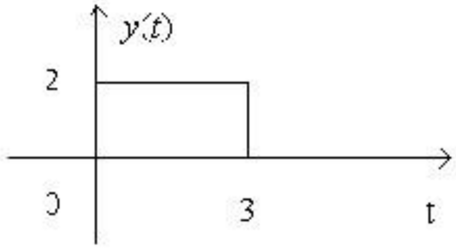


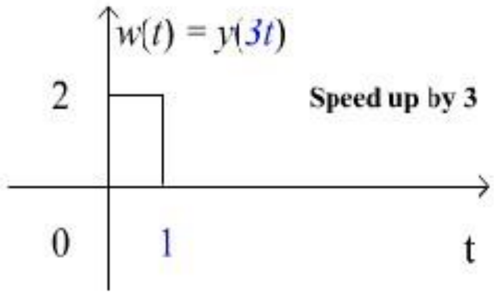
The period of $x(t)$ is 2 and the period of $y(t)$ is 1,

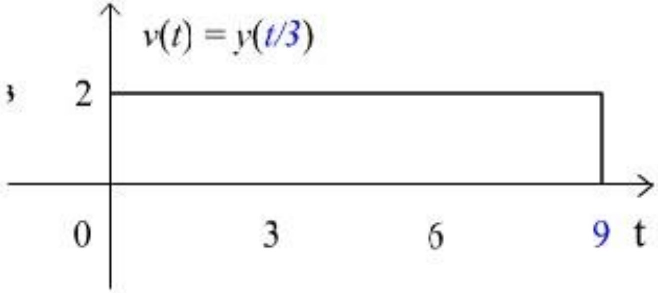
Time scaling Contd.

- Given $y(t)$,
  - find $w(t) = y(3t)$

  and $v(t) = y(t/3)$.





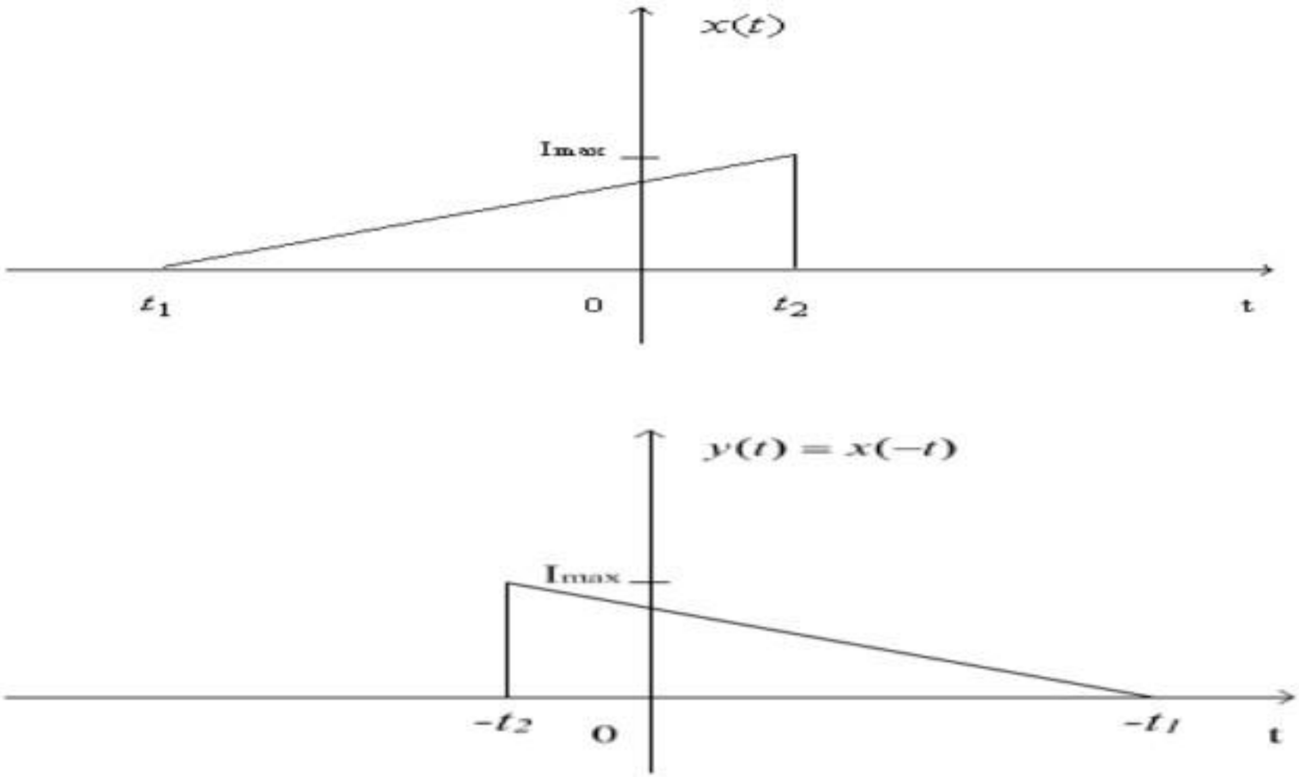


Time Reversal

- Time reversal is also called time folding
- In Time reversal signal is reversed with respect to time i.e.

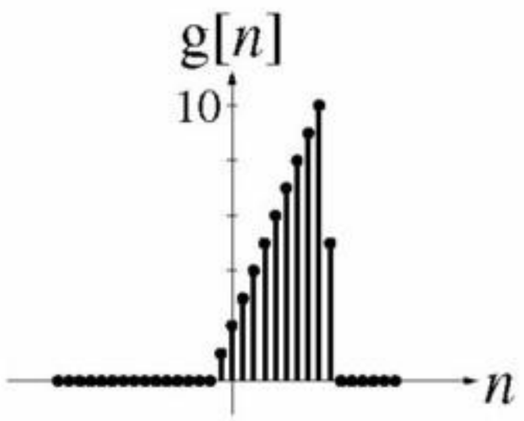
  $y(t) = x(-t)$ is obtained for the given function

Time reversal Contd.

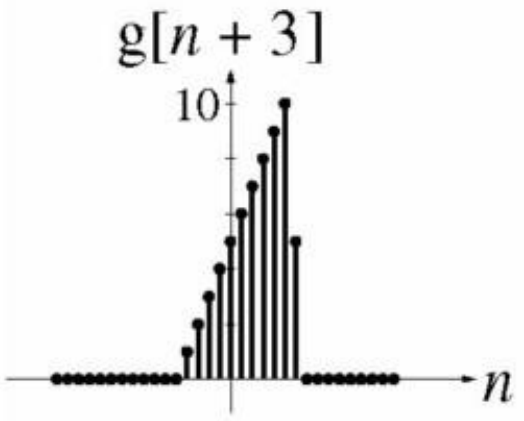


Operations of Discrete Time Functions

Timeshifting $n \to n + n_0$, n_0 an integer



n	$g[n]$
–1	1
0	2
1	3
2	4
3	5
4	6
5	7
6	8
7	9
8	10
9	5
10	0

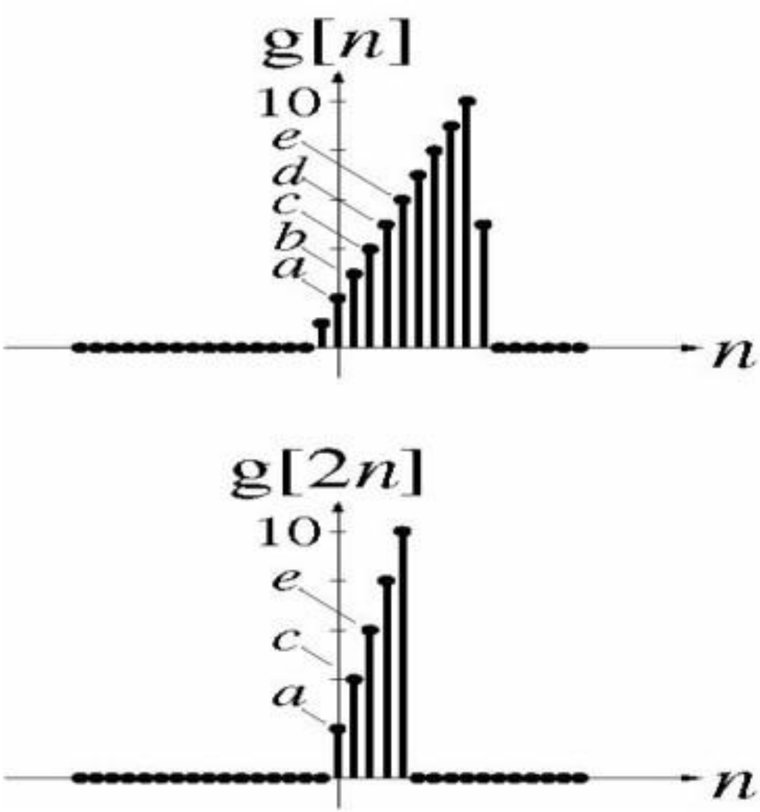


n	$n+3$	$g[n+3]$
–4	–1	1
–3	0	2
–2	1	3
–1	2	4
0	3	5
1	4	6
2	5	7
3	6	8
4	7	9
5	8	10
6	9	5
7	10	0

Operations of Discrete Functions Contd.

Scaling; Signal Compression

n • Kn K an integer > 1



n	$2n$	g[$2n$]
0	0	2
1	2	4
2	4	6
3	6	8
4	8	10

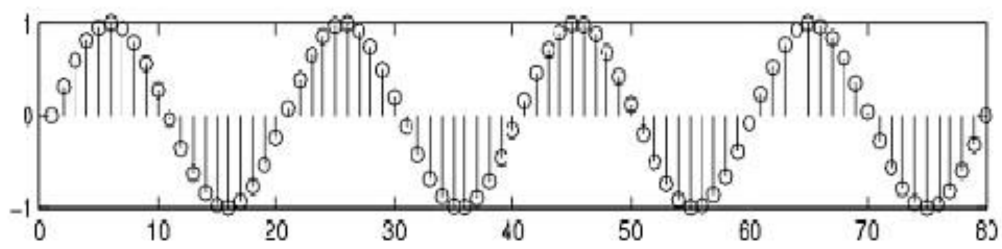
Classification of Signals

- Deterministic & Non Deterministic Signals
- Periodic & A periodic Signals
- Even & Odd Signals
- Energy & Power Signals

Deterministic & Non Deterministic Signals

Deterministic signals

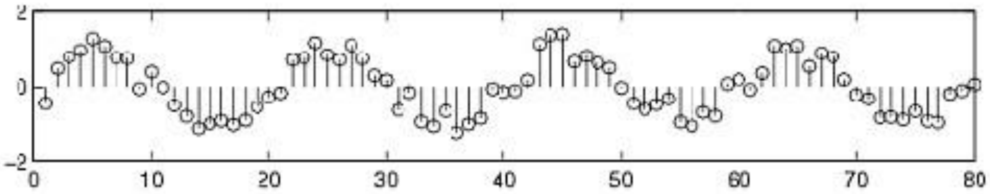
- **Behavior of these signals is predictable w.r.t time**
- **There is no uncertainty with respect to its value at any time.**
- **These signals can be expressed mathematically.**
  For example x(t) = sin(3t) is deterministic signal.



Deterministic & Non Deterministic Signals Contd.

Non Deterministic or Random signals

- Behavior of these signals is random i.e. not predictable w.r.t time.
- There is an uncertainty with respect to its value at any time.
- These signals can't be expressed mathematically.
- For example Thermal Noise generated is non deterministic signal.



Periodic and Non-periodic Signals

- Given x(t) is a continuous-time signal
- x (t) is periodic iff $x(t) = x(t+T_o)$ for any T and any integer n
- Example

  - $x(t) = A \cos(wt)$

  - $x(t+T_o) = A \cos[w\cdot\ t+T_o)] = A \cos(wt+wT_o) = A \cos(wt+2\cdot\ ) = A \cos(wt)$

  - Note: $T_o = 1/f_o$; $w\cdot\ 2\cdot\ f_o$

Periodic and Non-periodic Signals Contd.

- For non-periodic signals

$$x(t) \neq x(t+T_o)$$

- A non-periodic signal is assumed to have a period $T = \infty$
- Example of non periodic signal is an exponential signal

Important Condition of Periodicity for Discrete Time Signals

- A discrete time signal is periodic if

$$x(n) = x(n+N)$$

- For satisfying the above condition the frequency of the discrete time signal should be ratio of two integers

$$\text{i.e. } f_o = k/N$$

Sum of periodic Signals

- X(t) = x1(t) + X2(t)
- X(t+T) = x1(t+m_1T_1) + X2(t+m_2T_2)
- m_1T_1=m_2T_2 = T_o = Fundamental period
- Example: cos(t· /3)+sin(t· /4)
  - T1=(2·)/(· /3)=6; T2 =(2·)/(· /4)=8;
  - T1/T2=6/8 = ¾ = (rational number) = m2/m1
  - m_1T_1=m_2T_2 ·   Find m1 and m2·
  - 6.4 = 3.8 = 24 = T_o

Sum of periodic Signals - may not always be periodic!

$$x(t) \cdot \quad x_1(t) \cdot \quad x_2(t) \cdot \quad \cos t \cdot \quad \sin \sqrt{2t}$$

T1=(2•)/(1)= 2• ;   T2 =(2•)/(sqrt(2));

T1/T2= sqrt(2);

- Note: T1/T2 = sqrt(2) is an irrational number

- X(t) is aperiodic

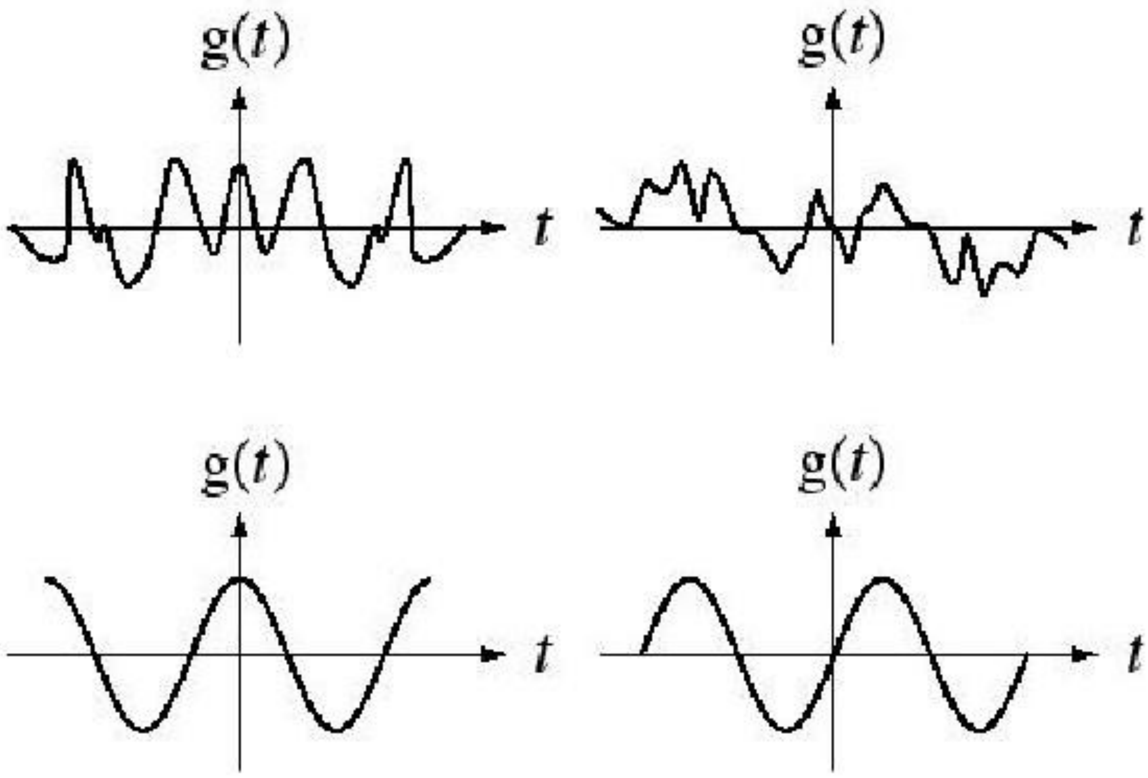
Even and Odd Signals

Even Functions

• • • g• • *t*•

Odd Functions

• • • • g• • *t*•



Even and Odd Parts of Functions

The **even part** of a function is $g_e(t) = \frac{g(t) + g(-t)}{2}$

The **odd part** of a function is $g_o(t) = \frac{g(t) - g(-t)}{2}$

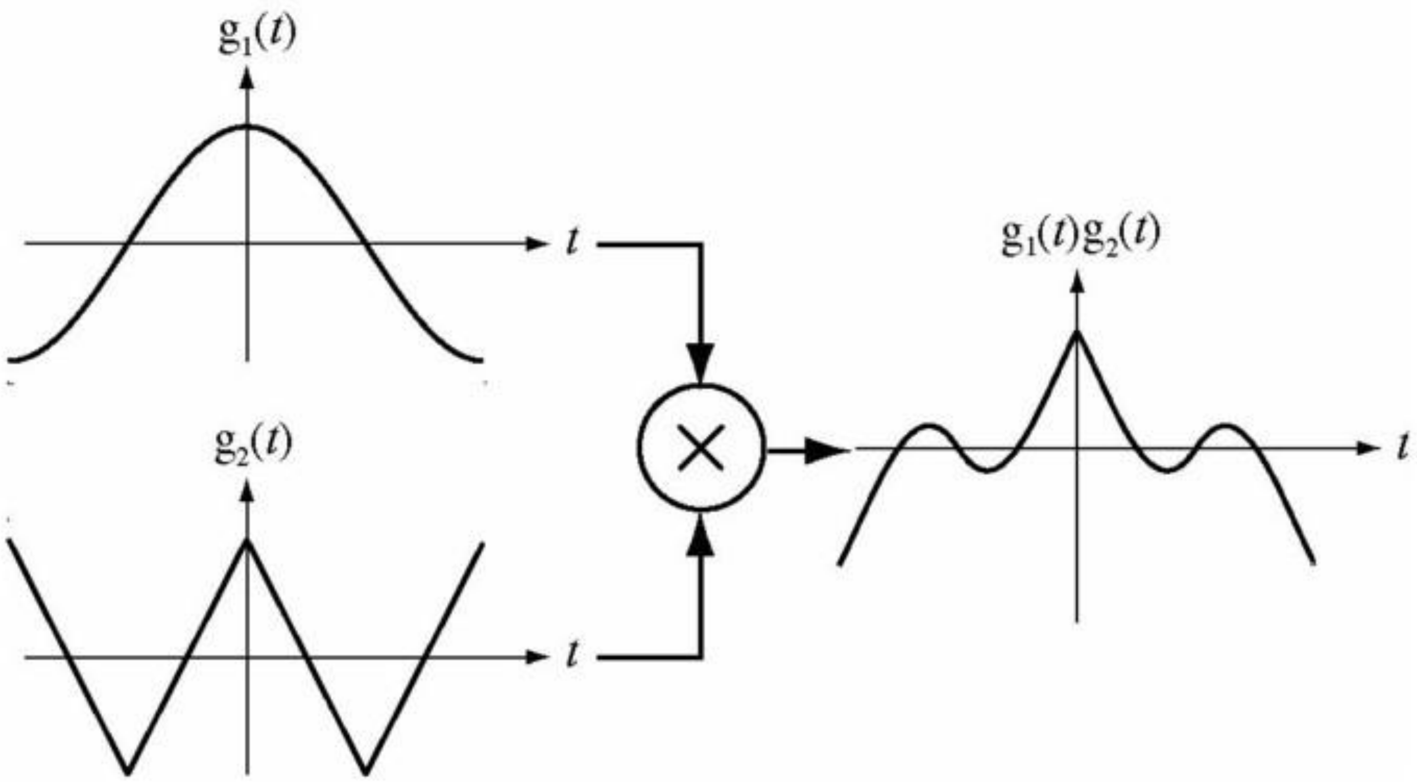
A function whose even part is zero, is odd and a function whose odd part is zero, is even.

Various Combinations of even and odd functions

Function type	Sum	Difference	Product	Quotient
Both even	Even	Even	Even	Even
Both odd	Odd	Odd	Even	Even
Even and odd	Neither	Neither	Odd	Odd

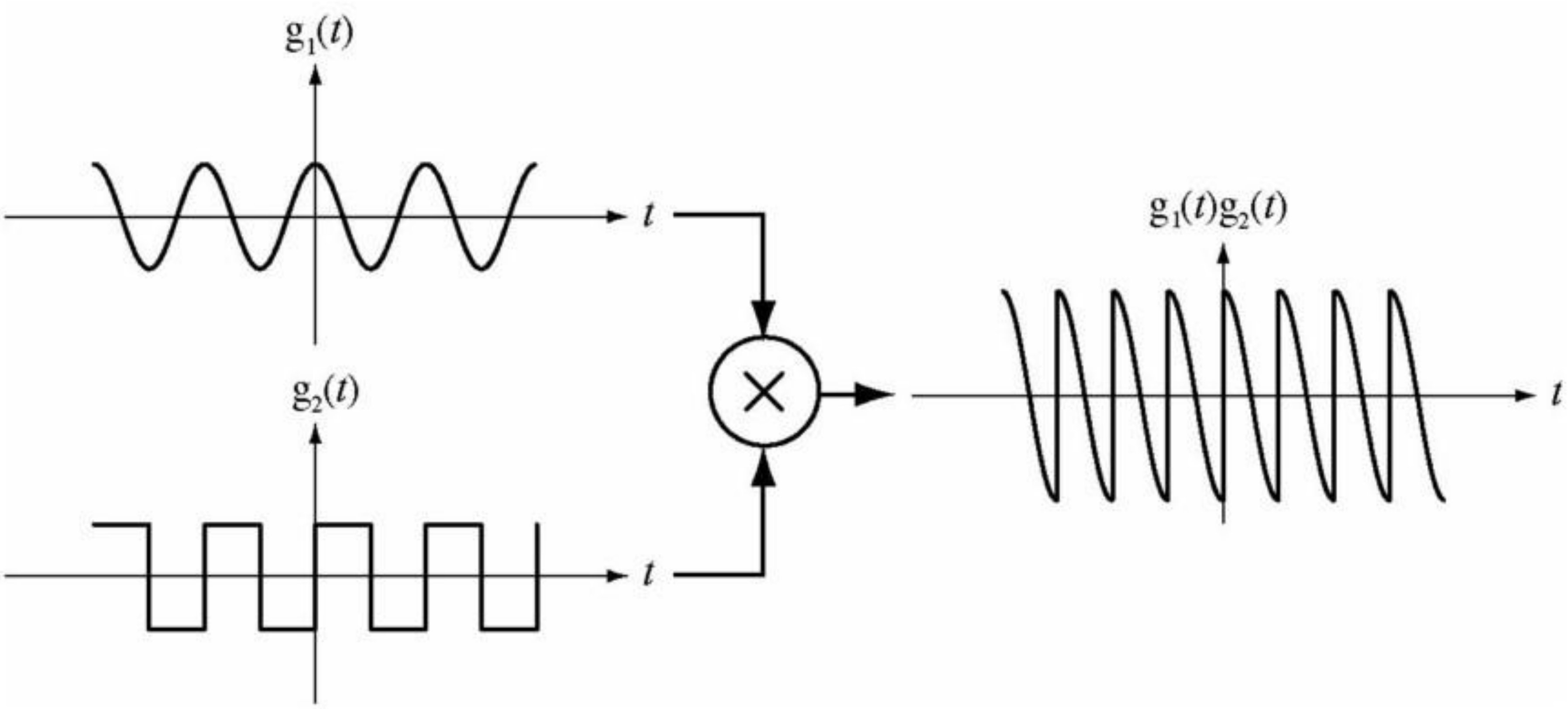
Product of Even and Odd Functions

Product of Two Even Functions



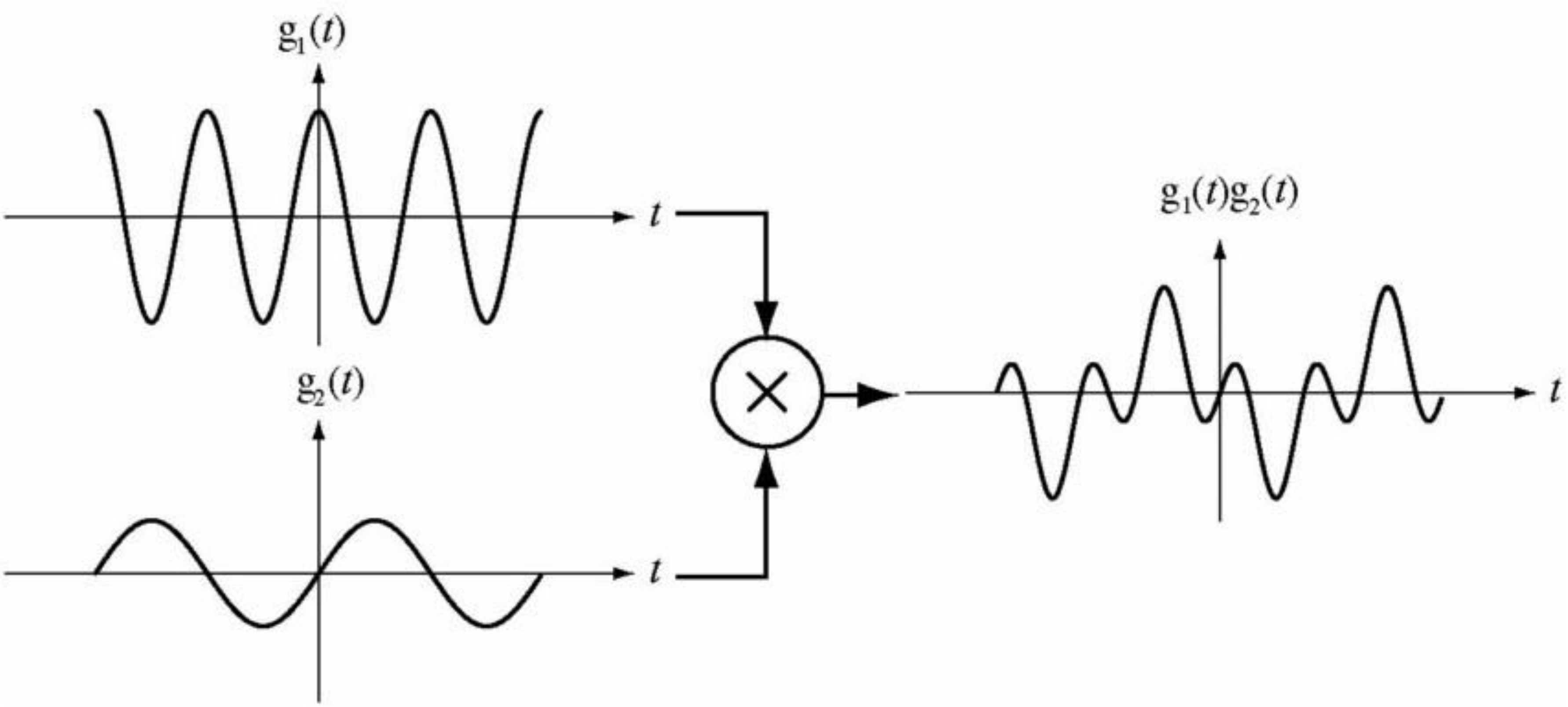
Product of Even and Odd Functions Contd.

Product of an Even Function and an Odd Function



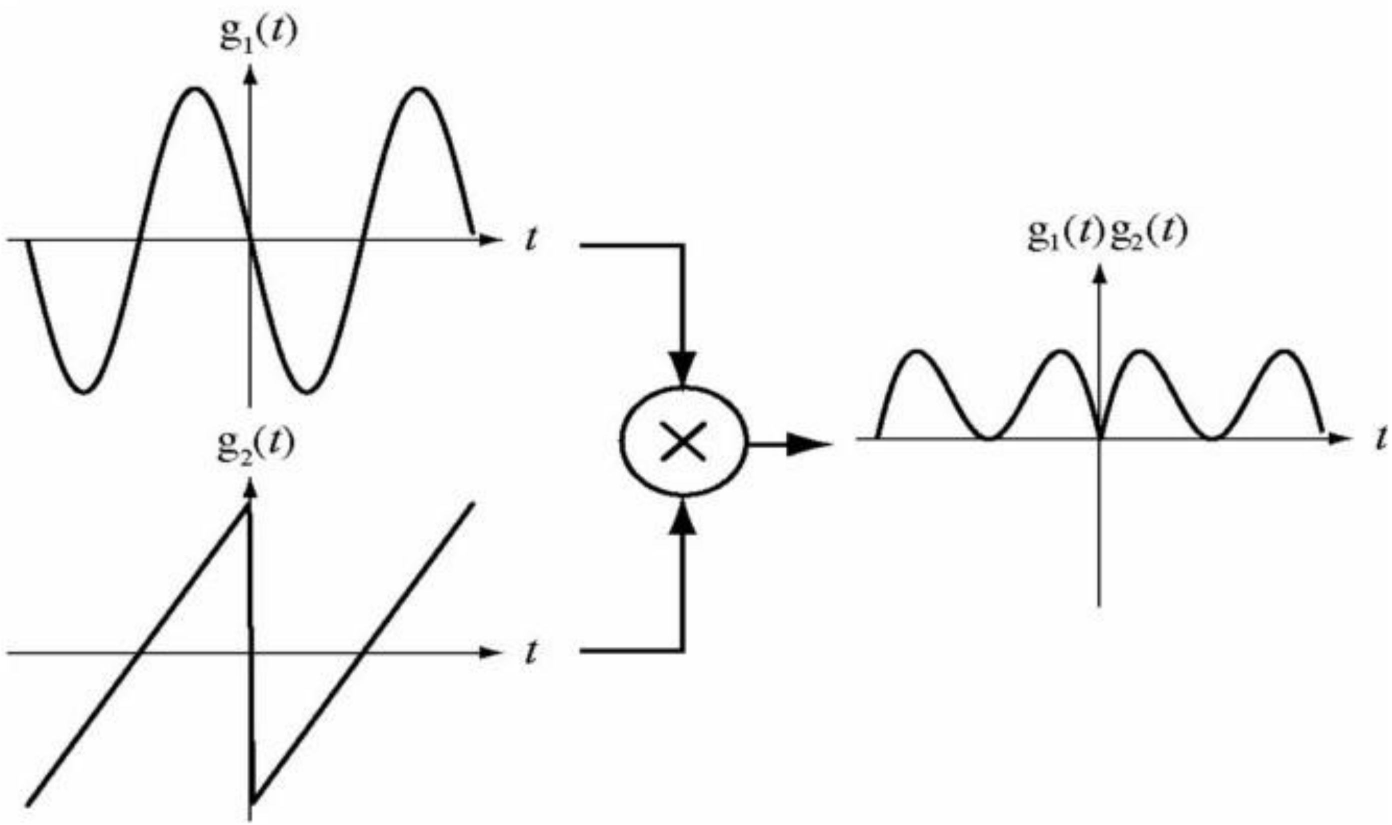
Product of Even and Odd Functions Contd.

Product of an Even Function and an Odd Function



Product of Even and Odd Functions Contd.

Product of Two Odd Functions



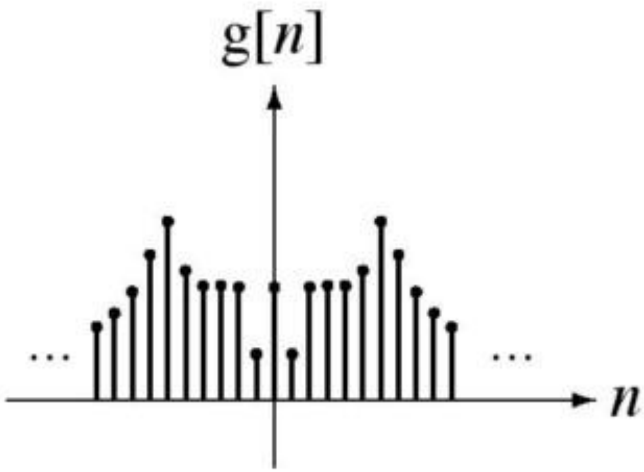
Derivatives and Integrals of Functions

Function type	Derivative	Integral
Even	Odd	Odd + constant
Odd	Even	Even

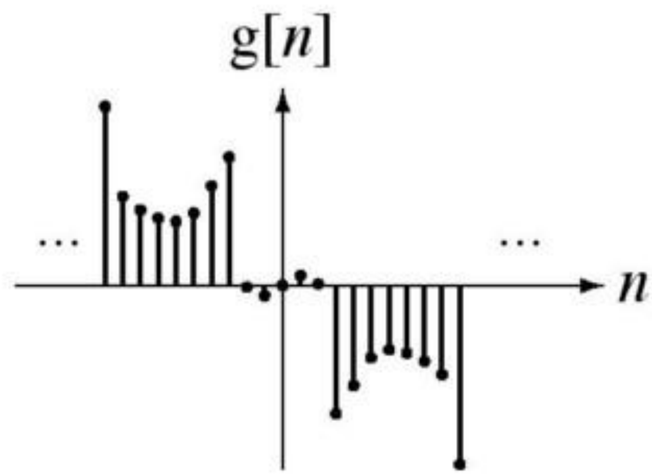
Discrete Time Even and Odd Signals

$$g[n] = g[-n] \qquad\qquad g[n] = -g[-n]$$

Even Function



Odd Function



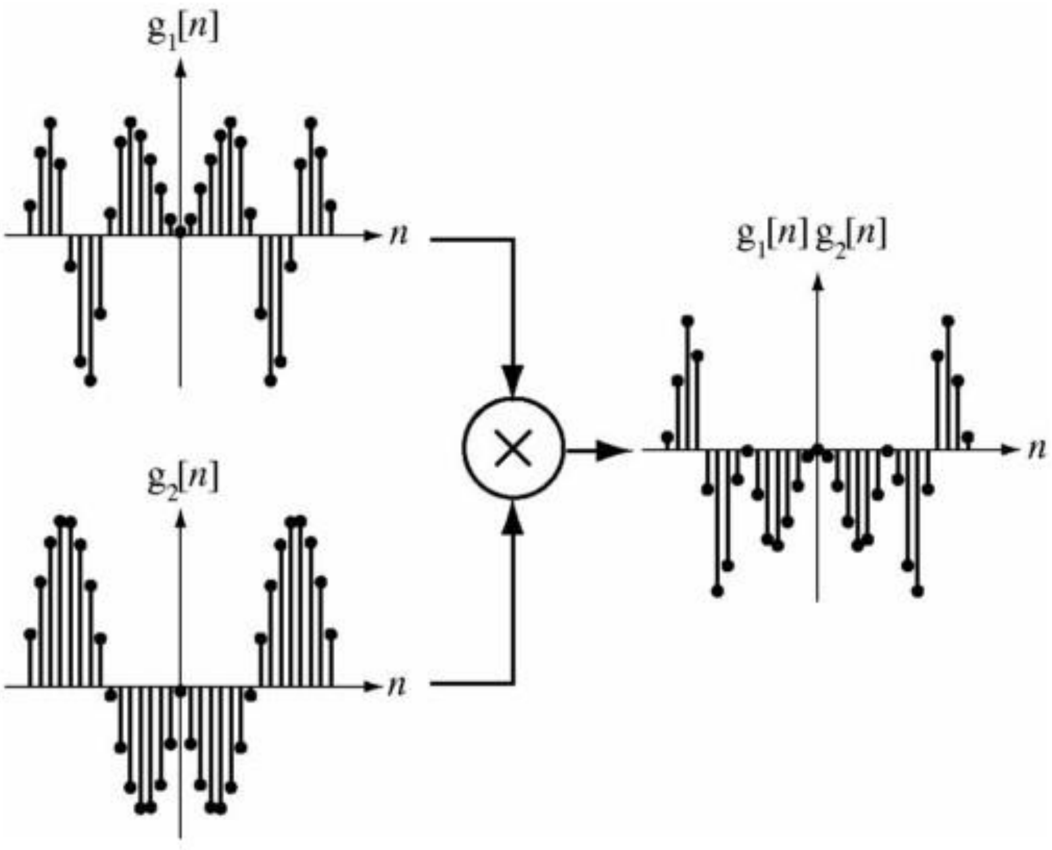
$$g_e[n] = \frac{g[n] + g[-n]}{2} \qquad\qquad g_o[n] = \frac{g[n] - g[-n]}{2}$$

Combination of even and odd function for DT Signals

Function type	Sum	Difference	Product	Quotient
Both even	Even	Even	Even	Even
Both odd	Odd	Odd	Even	Even
Even and odd	Even or Odd	Even or odd	Odd	Odd

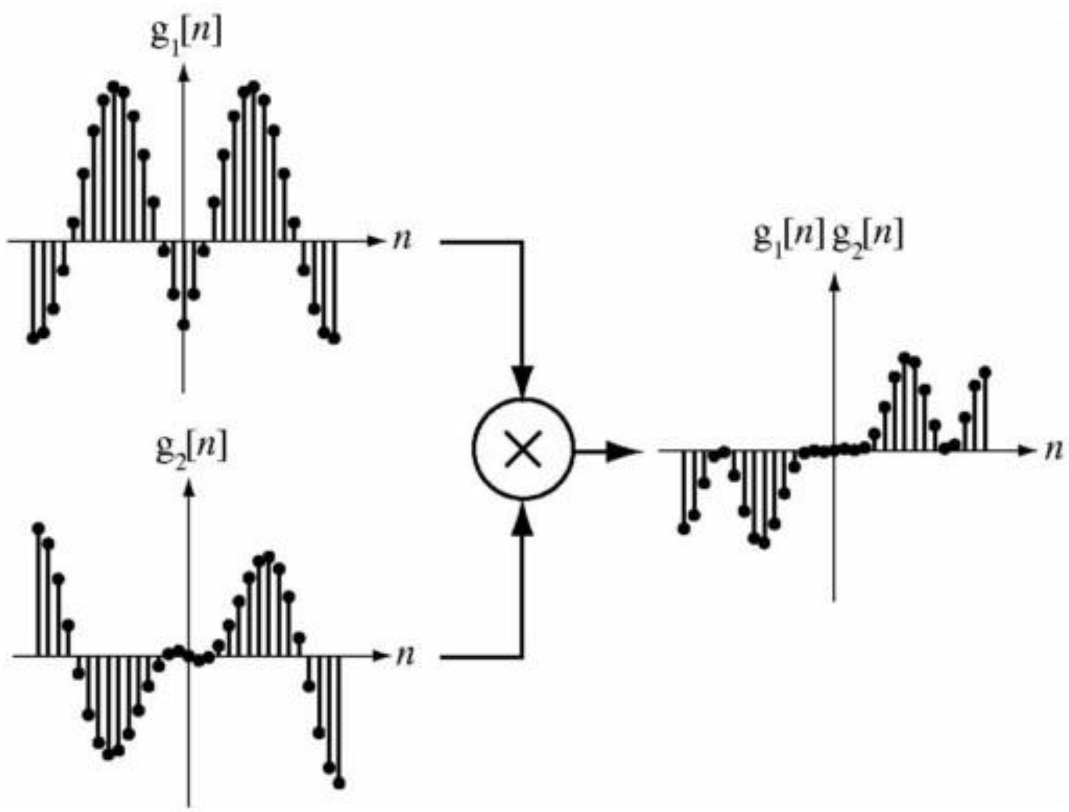
Products of DT Even and Odd Functions

Two Even Functions



Products of DT Even and Odd Functions Contd.

An Even Function and an Odd Function



Proof Examples

- Prove that product of two even signals is even.

Change t· -t

$x\ t)\cdot\ x\ t)\cdot_1\ x(t)\cdot\ _2$

$x(\cdot\ t)\cdot\ x(\cdot_1 t)\cdot\ x(\cdot\ t_2)\cdot$

$x_1 t)\cdot\ x\ t_2)\ \cdot\ x(t)$

- Prove that product of two odd signals is odd.

- What is the product of an even signal and an odd signal? Prove it!

$x(t)\ \cdot\ x_1(t)\ \cdot\ x_2(t)\ \cdot$

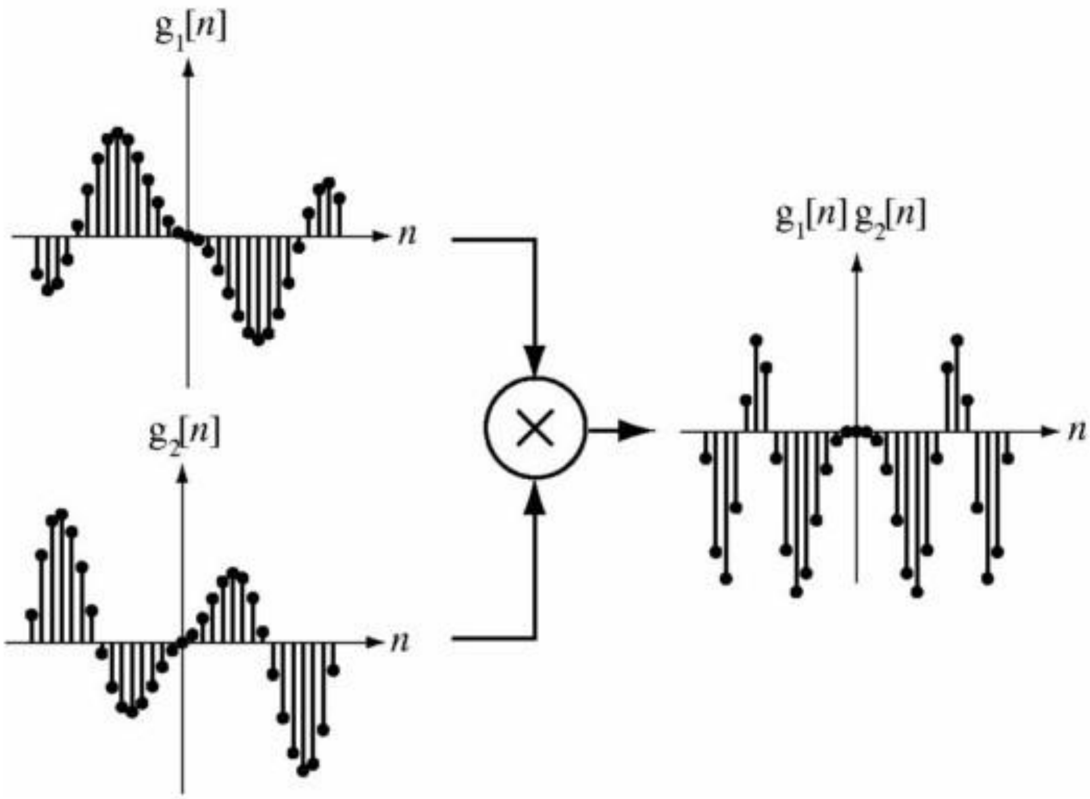
$x(\cdot\ t)\cdot\ x(\cdot_1 t)\cdot\ x(\cdot\ t_2)\cdot$

$x_1(t)\cdot\ \cdot\ x_2(t)\cdot\ \cdot\ x(t)\cdot$

$x(\cdot\ t)\ \cdot\ Even$

Products of DT Even and Odd Functions Contd.

Two Odd Functions



Energy and Power Signals

Energy Signal

- A signal with finite energy and zero power is called Energy Signal i.e.for energy signal

$$0<E<\infty \text{ and } P=0$$

- Signal energy of a signal is defined as the *area under the square of the magnitude of the signal*.

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- The units of signal energy depends on the unit of the signal.

Energy and Power Signals Contd.

Power Signal

- Some signals have infinite signal energy. In that caseit is more convenient to deal with **average signal power.**
- For power signals

$$0<P<\infty \text{ and } E = \infty$$

- Average power of the signal is given by

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

Energy and Power Signals Contd.

- For a periodic signal x(*t*) the average signal power is

$$P_x = \frac{1}{T}\int_T |x(t)|^2 \, dt$$

- *T* is any period of the signal.
- Periodic signals are generally power signals.

Signal Energy and Power for DT Signal

•A discrtet time signal with finite energy and zero power is called Energy Signal i.e.for energy signal

$0<E<\infty$ and $P = 0$

•The **signal energy** of a for a discrete time signal x[*n*] is

$$E_x = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

Signal Energy and Power for DT Signal Contd.

The average signal power of a discrete time power signal x[*n*] is

$$P_x = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \left| x[n] \right|^2$$

For a periodic signal x[*n*] the average signal power is

$$P_x = \frac{1}{N} \sum_{n=\langle N \rangle} \left| x[n] \right|^2$$

- The notation $\sum_{n=\langle N \rangle}$ means the sum over any set of consecutive n's exactly N in length.

What is System?

- Systems process input signals to produce output signals
- A system is combination of elements that manipulates one or more signals to accomplish a function and produces some output.

Examples of Systems

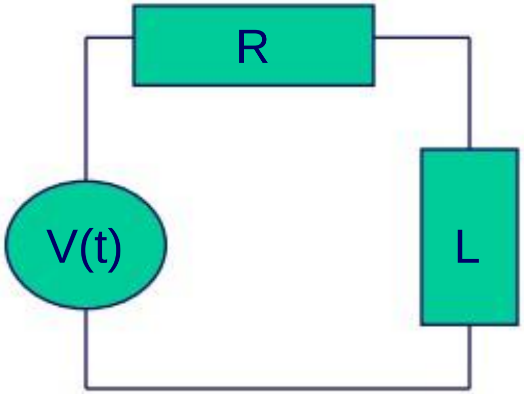
- A circuit involving a capacitor can be viewed as a system that transforms the source voltage (signal) to the voltage (signal) across the capacitor
- A communication system is generally composed of three sub-systems, the transmitter, the channel and the receiver. The channel typically attenuates and adds noise to the transmitted signal which must be processed by the receiver
- Biomedical system resulting in biomedical signal processing
- Control systems

System - Example

- Consider an RL series circuit
  - Using a first order equation:

$$V_L(t) = L\frac{di(t)}{dt}$$

$$V(t) = V_R + V_L(t) = i(t) \cdot R + L\frac{di(t)}{dt}$$



Mathematical Modeling of Continuous Systems

Most continuous time systems represent how continuous signals are transformed via **differential equations**.

E.g. RC circuit

System indicating velocity

$$\frac{dv_c(t)}{dt} \cdot \frac{1}{RC} v_c(t) \cdot \frac{1}{RC} v_s(t)$$

$$m \frac{dv(t)}{dt} \cdot \cdot v\, t) \cdot f(t)$$

Mathematical Modeling of Discrete Time Systems

Most discrete time systems represent how discrete signals are transformed via **difference equations**

e.g. bank account, discrete car velocity system

$$y[n] = 1.01y[n-1] + x[n]$$

$$v[n] = \frac{m}{m+\alpha\Delta} v[n-1] + \frac{\Delta}{m+\alpha\Delta} f[n]$$

Order of System

- Order of the Continuous System is the highest power of the derivative associated with the output in the differential equation
- For example the order of the system shown is 1.

$$m\frac{dv(t)}{dt} \cdot \quad \cdot \; v\; t) \cdot \; f(t)$$

Order of System Contd.

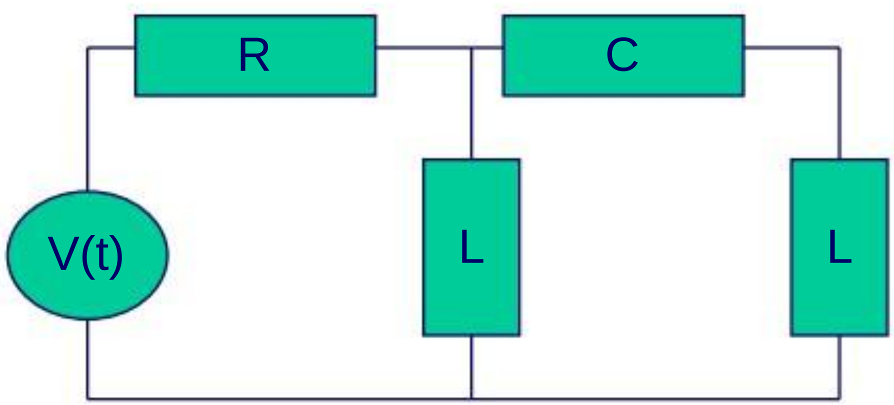
- Order of the Discrete Time system is the highest number in the difference equation by which the output is delayed
- For example the order of the system shown is 1.

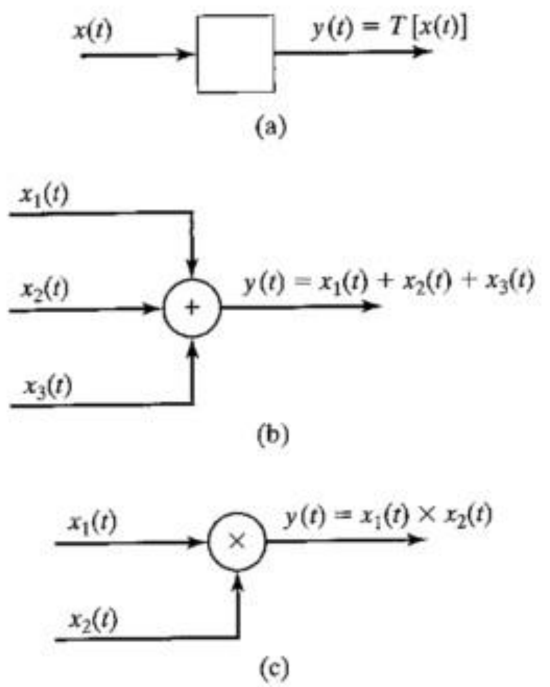
$$y[n] = 1.01y[n-1] + x[n]$$

Interconnected Systems

- Parallel
- Serial (cascaded)
- Feedback

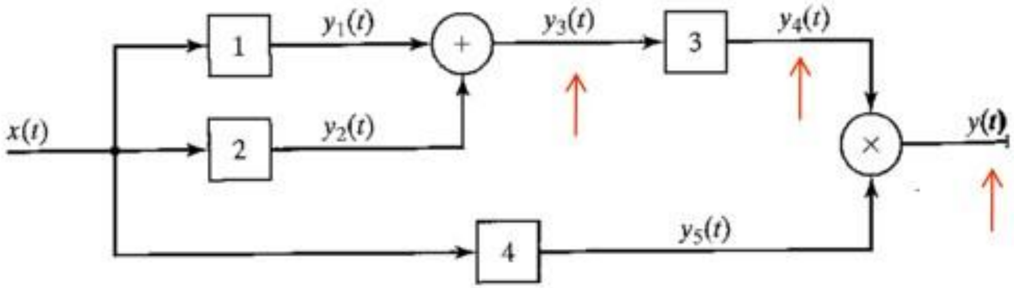
notes





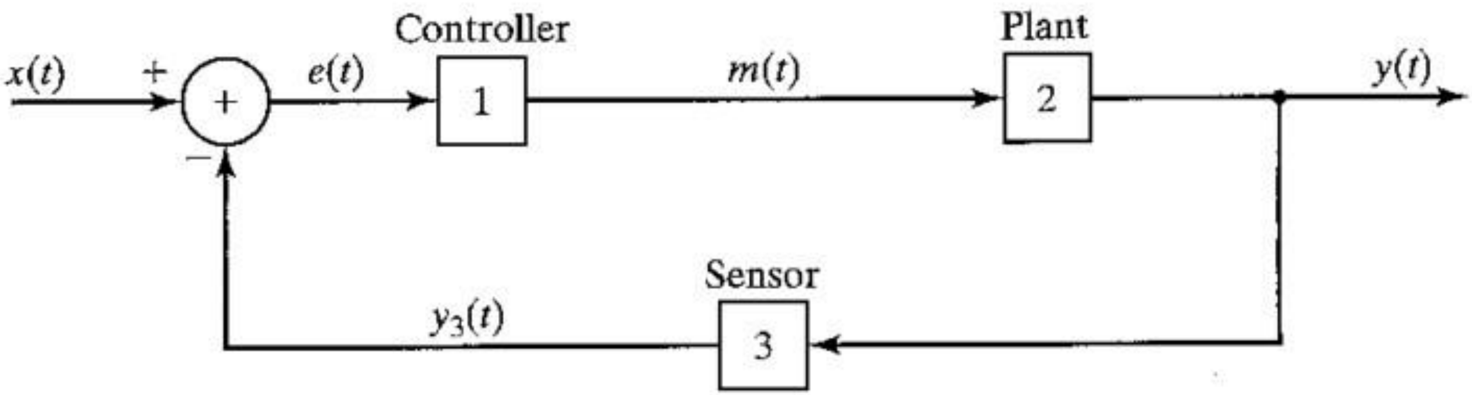
Interconnected System Example

- Consider the following systems with 4 subsystem
- Each subsystem transforms it input signal
- The result will be:
  - y3(t)=y1(t)+y2(t)=T1[x(t)]+T2[x(t)]
  - y4(t)=T3[y3(t)]= T3**(**T1[x(t)]+T2[x(t)])
  - y(t)= y4(t)* y5(t)= T3**(**T1[x(t)]+T2[x(t)])* T4[x(t)]



Feedback System

- Used in automatic control
  - e(t)=x(t)-y3(t)= x(t)-T3[y(t)]=
  - y(t)= T2[m(t)]=T2(T1[e(t)])
  - ∙ y(t)=T2(T1[x(t)-y3(t)])= T2(T1([x(t)] - T3[y(t)])) =
  - =T2(T1([x(t)] -T3[y(t)]))



Types of Systems

- Causal & Anticausal
- Linear & Non Linear
- Time Variant &Time-invariant
- Stable & Unstable
- Static & Dynamic
- Invertible & Inverse Systems

Causal & Anticausal Systems

- Causal system : A system is said to be *causal* if the present value of the output signal depends only on the present and/or past values of the input signal.
- Example: $y[n]=x[n]+1/2x[n-1]$

Causal & Anticausal Systems Contd.

- Anticausal system : A system is said to be *anticausal* if the present value of the output signal depends only on the future values of the input signal.
- Example: $y[n]=x[n+1]+1/2x[n-1]$

Linear & Non Linear Systems

- A system is said to be linear if it satisfies the principle of superposition
- For checking the linearity of the given system, firstly we check the response due to linear combination of inputs
- Then we combine the two outputs linearly in the same manner as the inputs are combined and again total response is checked
- If response in step 2 and 3 are the same,the system is linear othewise it is non linear.

Time Invariant and Time Variant Systems

- A system is said to be *time invariant* if a time delay or time advance of the input signal leads to a identical time shift in the output signal.

$$y_i(t) \cdot H x(t \cdot t_0)\}$$

$$\cdot H\{S^{t0}x(t)\}\} \cdot HS^{t0} x(t)\}$$

$$y_0(t) \cdot S^{t0}y(t)\}$$

$$\cdot S^{t0}\{H x(t)\}\} \cdot S^{t0} H x(t)\}$$

Stable & Unstable Systems

- A system is said to be *bounded-input bounded-output stable* (BIBO stable) iff every bounded input results in a bounded output.

i.e.

$$\cdot \; t \;\; |x \; t)| \cdot \; M_x \cdot \; \cdot \;\; \cdot \; \cdot \; t \;\; |y \; t)| \cdot \; M_y \cdot \; \cdot$$

Stable & Unstable Systems Contd.

Example

- $y[n]=1/3(x[n]+x[n-1]+x[n-2])$

$$y[n] = \frac{1}{3}\left|x[n] + x[n-1] + x[n-2]\right|$$

$$\le \frac{1}{3}(|x[n]| + |x[n-1]| + |x[n-2]|)$$

$$\le \frac{1}{3}(M_x + M_x + M_x) = M_x$$

Stable & Unstable Systems Contd.

Example: The system represented by

$$y(t) = A\,x(t) \text{ is unstable ; } A>1$$

Reason: let us assume x(t) = u(t), then at every instant u(t) will keep on multiplying with A and hence it will not be bonded.

Static & Dynamic Systems

- A static system is memoryless system
- It has no storage devices
- its output signal depends on present values of the input signal
- For example

$$i(t) = \frac{1}{R} v(t)$$

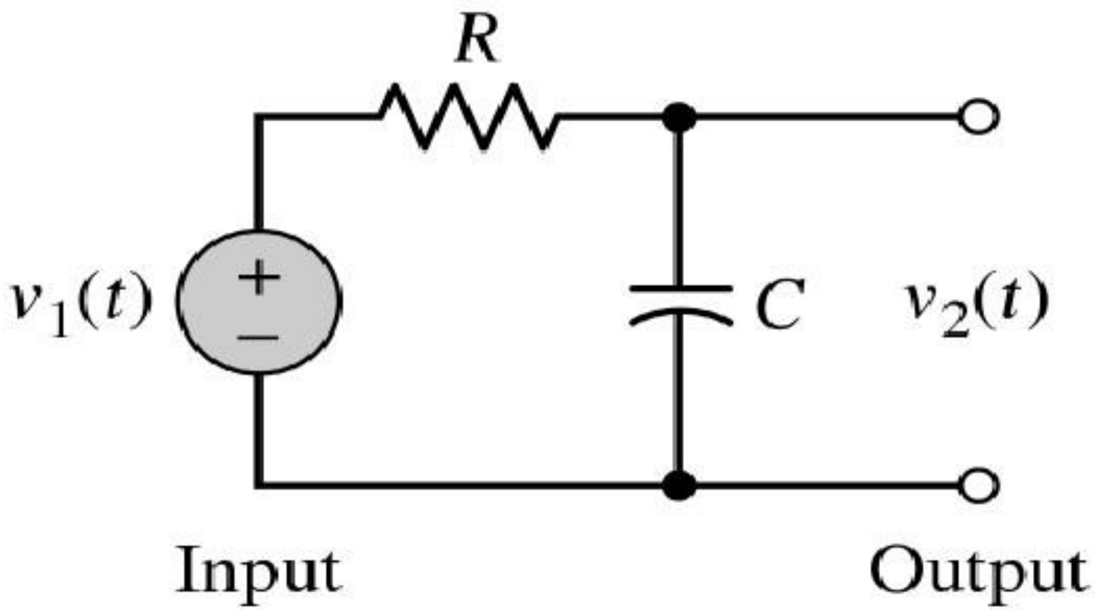
Static & Dynamic Systems Contd.

- A dynamic system possesses memory
- It has the storage devices
- A system is said to possess *memory* if its output signal depends on past values and future values of the input signal

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau$$

$$y[n] = x[n] + x[n-1]$$

Example: Static or Dynamic?



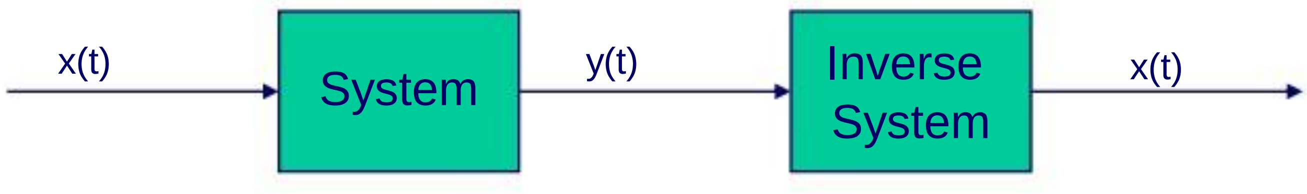
Example: Static or Dynamic?

Answer:

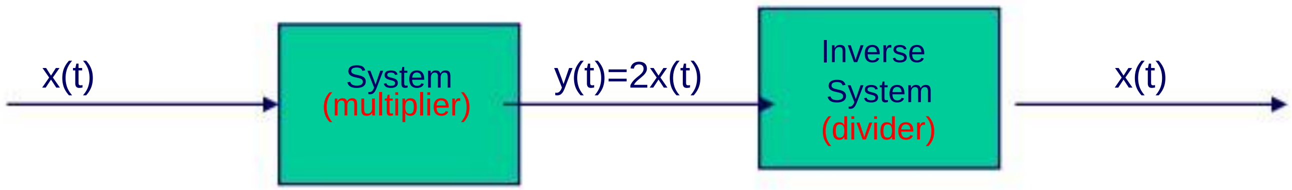
- The system shown above is RC circuit
- R is memoryless
- C is memory device as it stores charge because of which voltage across it can‟t change immediately
- Hence given system is dynamic or memory system

Invertible & Inverse Systems

- If a system is invertible it has an Inverse System



- Example: $y(t)=2x(t)$
  - System is invertible · must have inverse, that is:
  - For any $x(t)$ we get a distinct output $y(t)$
  - Thus, the system must have an Inverse
    - $x(t)=1/2\ y(t)=z(t)$



LTI Systems

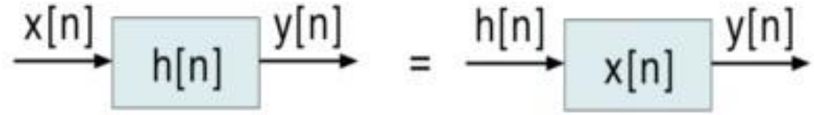
- LTI Systems are *completely characterized* by its unit sample response
- The output of *any* LTI System is a convolution of the input signal with the unit-impulse response, *i.e.*

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Properties of Convolution

Commutative Property

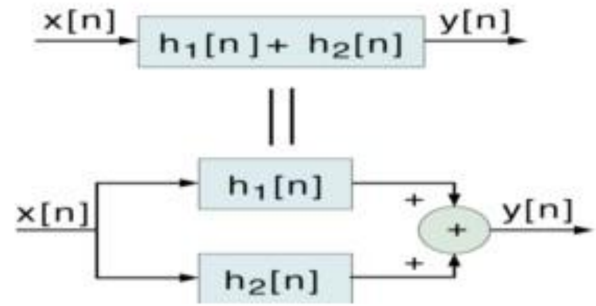
$$x[n]*h[n]\cdot \quad h[n]*x[n]$$



Distributive Property

$$x[n]*(h_1[n]\cdot \quad h_2[n]) \quad \cdot$$

$$(x[n]*h_1[n])\cdot \quad (x[n]*h_2[n])$$

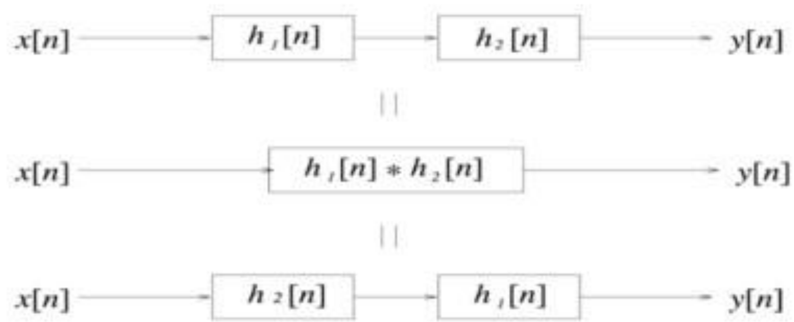


Associative Property

$$x[n]*h_1[n]*h_2[n] \quad \cdot$$

$$(x[n]*h_1[n])*h_2[n] \quad \cdot$$

$$(x[n]*h_2[n])*h_1[n]$$



Useful Properties of (DT) LTI Systems

- **Causality:** $h[n] = 0 \quad n < 0$

- **Stability:** $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

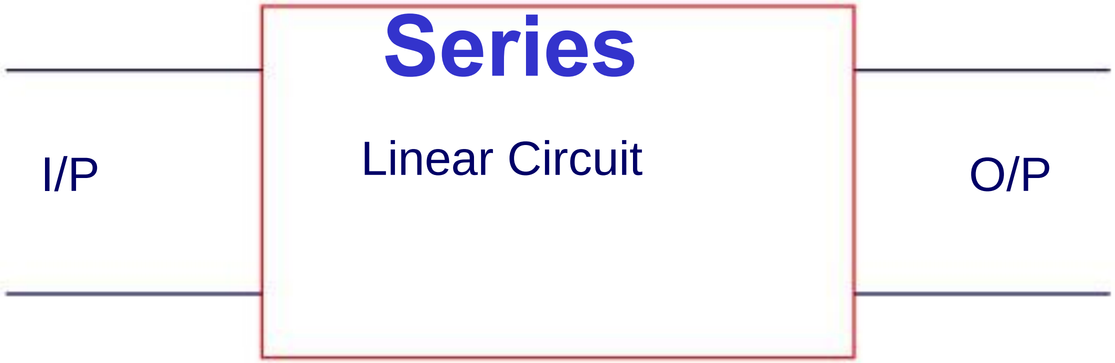
Bounded Input ↔ Bounded Output

$$\text{for } |x[n]| \le x_{max} < \infty$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \le x_{max} \left| \sum_{k=-\infty}^{\infty} h[n-k] \right| < \infty$$

Periodic Functions and Fourier Series

The Fourier Series



Sinusoidal Inputs **OK**

Nonsinusoidal Inputs

Nonsinusoidal Inputs Sinusoidal Inputs

Fourier Series

The Fourier

Joseph Fourier
1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

The Fourier Series

Fourier proposed in 1807

A periodic waveform *f(t)* could be broken down into an *infinite series of simple sinusoids* which, when added together, would construct the *exact form* of the original waveform.

Consider the periodic function

$$f(t) = f(t + nT) \quad ; n = \pm 1, \pm 2, \pm 3, \cdots$$

T = Period, the smallest value of T that satisfies the above Equation.

The Fourier Series

The expression for a ***Fourier Series*** is

$$f(t) = a_0 + \sum_{n=1}^{N} a_n \cos n\omega_0 t + \sum_{n=1}^{N} b_n \sin n\omega_0 t$$

a_0, a_n, and b_n are real and are called **Fourier Trigonometric Coefficients** and $\omega_0 = \frac{2\pi}{T}$

Or, alternative form

$$f(t) = C_0 + \sum_{n=1}^{N} C_n \cos(n\omega_0 t + \theta_n)$$

$C_0 = a_0$ and C_n are the **Complex Coefficients**

Fourier Series = a finite sum of harmonically related sinusoids

The Fourier Series

$$f(t) = C_0 + \sum_{n=1}^{N} C_n \cos(n\omega_0 t + \theta_n)$$

C_0 is the average (or DC) value of *f(t)*

For $n = 1$ the corresponding sinusoid is called ***the fundamental***

$$C_1 \cos(\omega_0 t + \theta_1)$$

For $n = k$ the corresponding sinusoid is called ***the k_{th} harmonic term***

$$C_k \cos(k\omega_0 t + \theta_k)$$

Similarly, ω_0 is call the ***fundamental frequency***

$k\omega_0$ is called the ***k_{th} harmonic frequency***

The Fourier Series

Definition

$N\cdot \quad \cdot$

A ***Fourier Series*** is an accurate representation of a periodic signal and consists of the sum of sinusoids at the fundamental and harmonic frequencies.

The waveform *f(t)* depends on the ***amplitude*** and ***phase*** of every harmonic components, and we can generate any non-sinusoidal waveform by an appropriate combination of sinusoidal functions.

The Fourier Series (Dirichlet's Conditions)

To be described by the Fourier Series the waveform *f(t)* must satisfy the following mathematical properties:

1. *f(t)* is a ***single-value function*** except at possibly a finite number of points.

2. The integral $\int_{t_0}^{t_0+T} |f(t)|dt < \infty$ for any t_0.

3. *f(t)* has a finite number of ***discontinuities*** within the period *T*.
4. *f(t)* has a finite number of ***maxima*** and ***minima*** within the period *T*.

In practice, *f(t)* = *v(t)* or *i(t)* so the above 4 conditions are always satisfied.

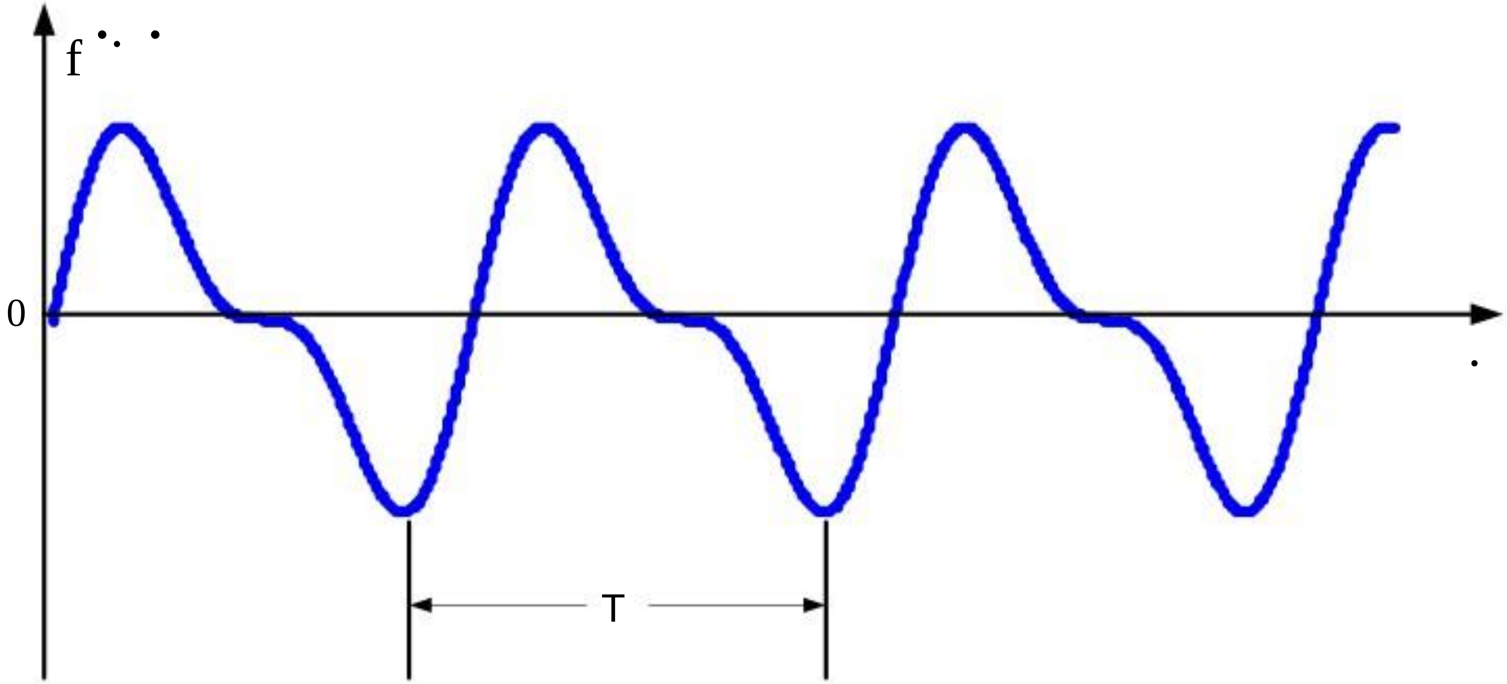
Periodic Functions

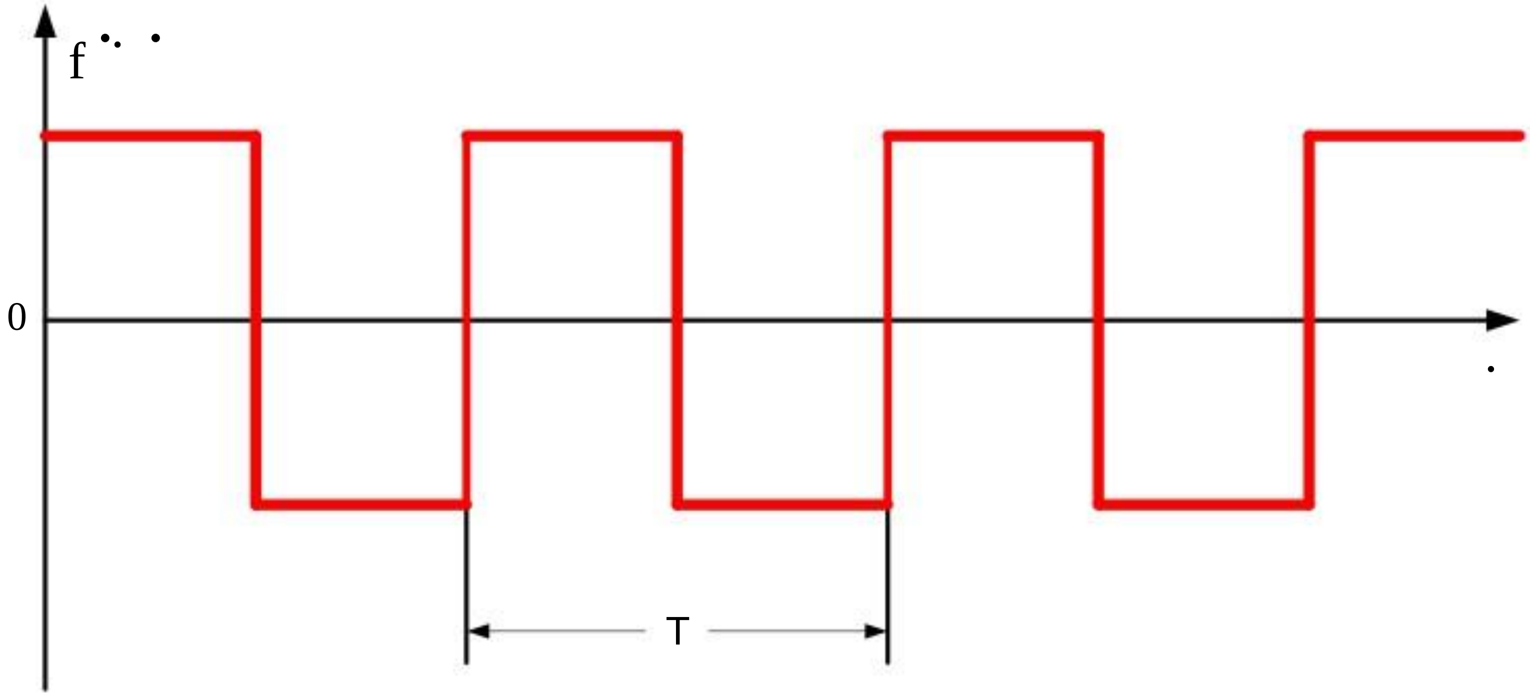
A function $f \cdot \cdot \cdot$ is periodic

if it is defined for all real $\cdot$

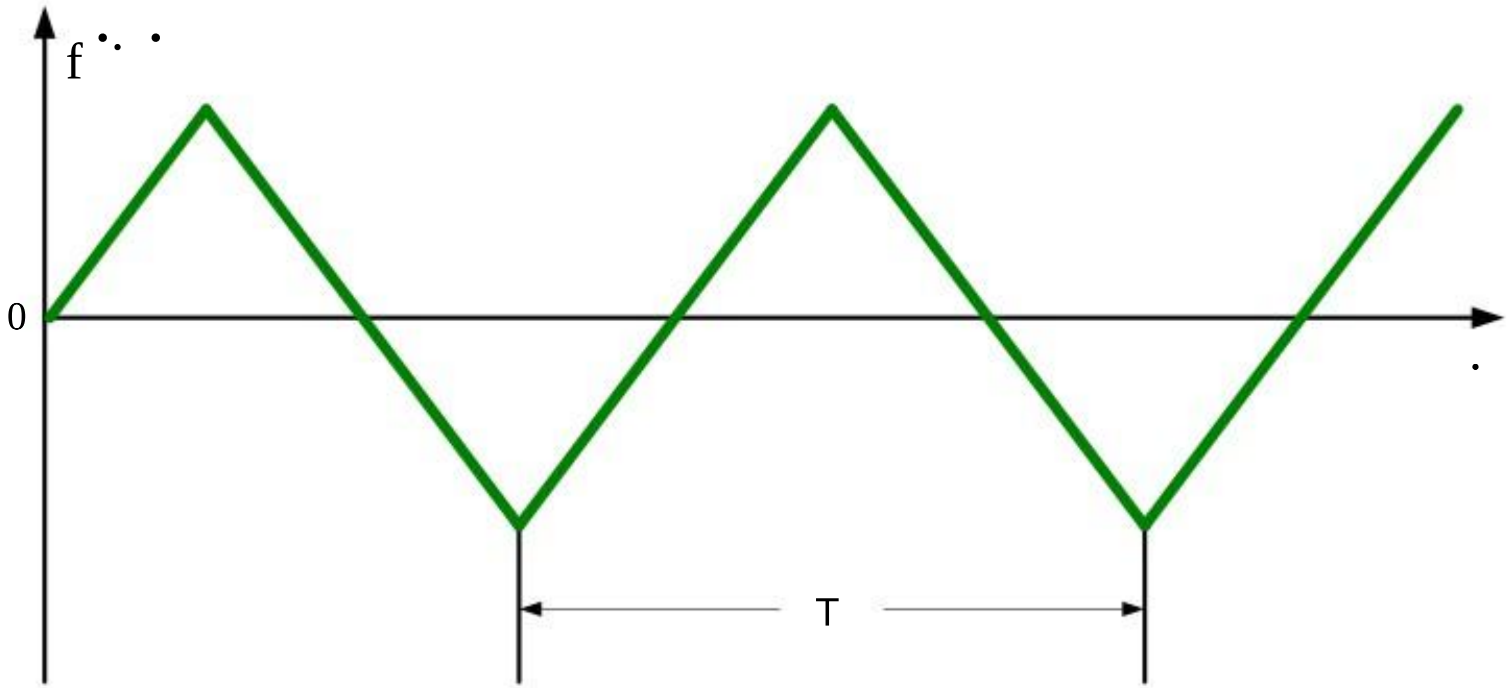
and if there is some positive number,

T such that $f \cdot \cdot \cdot T \cdot \cdot f \cdot \cdot \cdot$


f
0
T



f
0
T


f
0
T

Fourier Series

$f\cdot \;\; \cdot \;\; \cdot$ be a periodic function with period $2\cdot$

The function can be represented by a trigonometric series as:

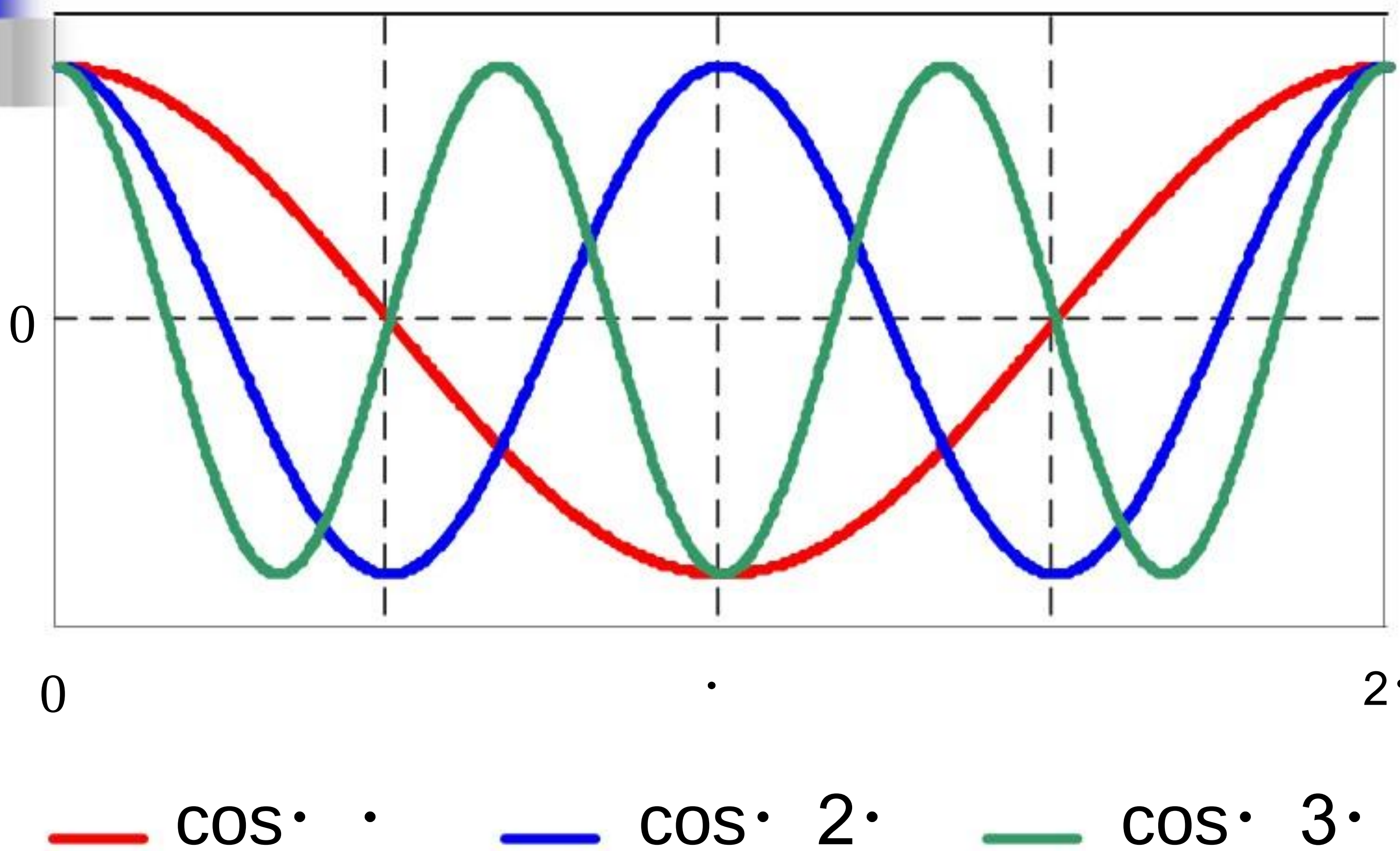
$$f\cdot \;\; \cdot \;\; \cdot \;\; \cdot_0 a \;\; \overset{\cdot}{\underset{n\cdot\ 1}{\bullet}} \;\; a_n \cos n\cdot \;\; \cdot \;\; \overset{\cdot}{\underset{n\cdot\ 1}{\bullet}} \;\; b_n \sin n\cdot$$

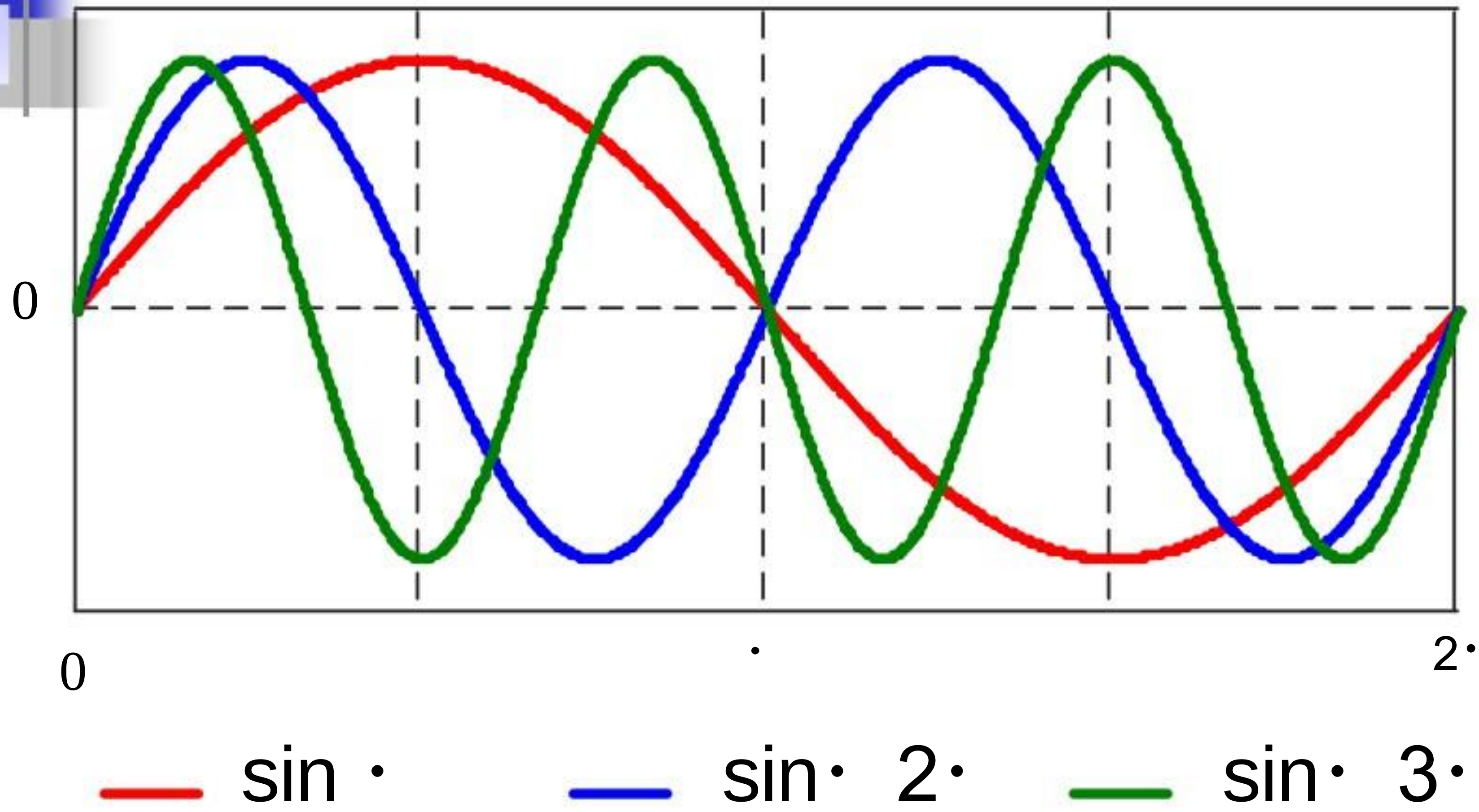
$$\cdot \;\; \cdot \;\; \cdot \;\; q_0 \;\cdot\; \mathop{\bullet}_{n\cdot 1}^{\cdot} a_n \cos n\cdot \;\; \cdot \;\; \mathop{\bullet}_{n\cdot 1}^{\cdot} b_n \sin n\cdot$$

What kind of trigonometric (series) functions are we talking about?

$\cos \cdot,\ \cos 2\cdot,\ \cos 3\cdot\ \cdot$ and

$\sin \cdot,\ \sin 2\cdot,\ \sin 3\cdot\ \cdot$


0
0
·
2·
cos· ·
cos· 2·
cos· 3·



0
0
·
2·
sin ·
sin· 2·
sin· 3·

We want to determine the coefficients,

a_n **and** b_n

Let us first remember some useful integrations.

- **cos*n*· cos*m*· *d*·**

· $\frac{1}{2}$ · cos· *n*·*m*· · *d*· · $\frac{1}{2}$ · cos· *n*· *m*· · *d*·

- *cos* ***n***· *cos* ***m*** ***d*** · 0 ***n*· *m***

- *cos* ***n***· *cos* ***m*** ***d*** · · ***n*· *m***

$$\cdot_{\cdot\,\cdot}^{\cdot}\ \sin n\cdot\ \cos m\cdot\ d\cdot$$

$$\cdot\ \frac{1}{2}\cdot_{\cdot\,\cdot}^{\cdot}\ \sin\cdot\ n\cdot\ m\ \cdot\ d\cdot\ \cdot\frac{1}{2}\cdot_{\cdot\,\cdot}^{\cdot}\ \sin\cdot\ n\cdot\ m\cdot\ \cdot\ d\cdot$$

$$\cdot_{\cdot\,\cdot}^{\cdot}\ sin\, n\cdot\ cos\, m\ d\ \cdot\ 0$$

for all values of *m*.

$\sin n\cdot \sin m\cdot d\cdot$

$\cdot\ \frac{1}{2}\cdot\ \cos\cdot\ n\cdot m\ d\cdot\ \ \cdot\ \frac{1}{2}\cdot\ \cos\cdot\ n\cdot\ m\cdot\ \cdot\ d\cdot$

- $\sin n\cdot \sin\ m\cdot\ d\cdot\ \cdot\ 0$ $\quad n\cdot\ m$

- $\sin n\cdot \sin\ m\cdot\ d\cdot\ \cdot\ \cdot$ $\quad n\cdot\ m$

Determine a_0

Integrate both sides of (1) from

· · to ·

$$\cdot \; f \cdot \; \cdot \; d\cdot \; \cdot$$

$$\cdot \; \cdot \; \cdot\, a_0 \; \cdot \; \cdot \; a_n \cos n\cdot \; \cdot \; \cdot \; b_n \sin n\cdot \; \cdot\, d\cdot$$

$$n\cdot 1 \qquad n\cdot 1$$

$$f \cdot \cdot \cdot \; d \cdot$$

$$\cdot \; \cdot \; a_0 d \cdot \; \cdot \; \cdot \; \cdot \; \cdot_{n \cdot 1} \cdot \; a_n \cos n \cdot \; \cdot \; d \cdot$$

$$\cdot \; \cdot \; \cdot_{n \cdot 1} \cdot \; b_n \sin n \cdot \; \cdot \; d \cdot$$

$$\cdot \; f \cdot \cdot \cdot \; d \cdot \cdot \; \cdot \; a_0 d \cdot \; \cdot \; 0 \cdot \; 0$$

$$\bullet f\cdot \cdot \cdot d\cdot \cdot 2_0 \cdot \cdot a_0 \cdot 0$$

$$a_0 \cdot \frac{1}{2\cdot} \bullet f\cdot \cdot \cdot d\cdot$$

a_0 **is the average (dc) value of the function,** $f\cdot \cdot \cdot$

You may integrate both sides of (1) from 0 to 2π instead.

$$\int_0^{2\pi} f(\theta)\, d\theta$$

$$= \int_0^{2\pi} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \right] d\theta$$

It is alright as long as the integration is performed over one period.

$$\int_0^{2\pi} f(\theta)\, d\theta$$

$$= \int_0^{2\pi} a_0\, d\theta + \int_0^{2\pi} \left(\sum_{n=1}^{\infty} a_n \cos n\theta \right) d\theta$$

$$+ \int_0^{2\pi} \left(\sum_{n=1}^{\infty} b_n \sin n\theta \right) d\theta$$

$$\int_0^{2} f(\theta)\, d\theta = \int_0^{2\pi} a_0\, d\theta = 0 \Rightarrow \quad 0$$

$$\bullet_0^{\cdot} f\cdot\cdot\ \cdot d\cdot\ \cdot\ 2\cdot a_0\ \cdot\ 0\cdot\ 0$$

$$a_0 \cdot \frac{1}{2\cdot} \bullet_0^{2\cdot} f\cdot\ \cdot\ \cdot\ d\cdot$$

Determine a_n

Multiply (1) by $\cos m\cdot$

and then Integrate both sides from

$\cdot\ \cdot$ to $\cdot$

$$\cdot\int_{\cdot\ \cdot}^{\cdot} f(\cdot)\cos m\cdot\ d\cdot$$

$$\cdot\ \cdot \int_{\cdot\ \cdot}^{\cdot}\left[a_0 \cdot \sum_{n\cdot 1}^{\cdot} a_n \cos n\cdot \cdot \sum_{n\cdot 1}^{\cdot} b_n \sin n\cdot\right]\cos m\cdot\ d\cdot$$

Let us do the integration on the right-hand-side one term at a time.

First term,

$$\bullet\, a_0 \cos m\cdot \; d\cdot \quad \cdot \; 0$$

Second term,

$$a_n \cos n\cdot \; \cos m\cdot \; d\cdot$$

$$n\cdot \; 1$$

Second term,

$$\cdot_{\cdot}^{\cdot} \; \cdot_{n\cdot 1}^{\cdot} \; a_n \cos n\cdot \; \cos m\cdot \; d\cdot \; \cdot \; a_{m}\cdot$$

Third term,

$$\cdot_{\cdot}^{\cdot} \; \cdot_{n\cdot 1}^{\cdot} \; b_n \sin n\cdot \; \cos m\cdot \; d\cdot \; \cdot \; 0$$

Therefore,

$$\overset{\cdot}{\underset{\cdot\ \cdot}{\bullet}}\ f\cdot \quad \cdot \quad \cdot \quad \cos m\cdot \ d\cdot \quad \cdot \quad a\cdot_{m}$$

$$a_m \cdot \frac{1}{\cdot} \overset{\cdot}{\underset{\cdot\ \cdot}{\bullet}}\ f\cdot \quad \cdot \quad \cdot \quad \cos m\cdot \ d\cdot m\cdot \ 1, 2, \cdot$$

Determine b_n

Multiply (1) by *sinm·*

and then Integrate both sides from

· · to ·

$$\cdot \; f\cdot \;\cdot\; \cdot \; sin\,\cdot \; d\cdot$$

$$\cdot \; \cdot \; a_0 \cdot \; \cdot \sum_{n\cdot 1} a_n \cos n\cdot \;\cdot\; \cdot \sum_{n\cdot 1} b_n \sin n\cdot \;\; sinm\cdot \; d\cdot$$

Let us do the integration on the right-hand-side one term at a time.

First term,

$$\bullet \int_{\cdot\,\cdot}^{\cdot} a_0 \sin m\cdot \; d\cdot \;\cdot\; 0$$

Second term,

$$\bullet \int_{\cdot\,\cdot}^{\cdot} \;\bullet_{n\cdot\,1}\; a_n \cos n\cdot \; \sin m\cdot \; d\cdot$$

Second term,

$$\cdot \quad \cdot \quad a_n \cos n \cdot \; \sin m \cdot \; d\cdot \; \cdot \; 0$$
$$n \cdot 1$$

Third term,

$$\cdot \quad \cdot \quad b_n \sin n \cdot \; \sin m \cdot \; d\cdot \; \cdot \; b_m^{\cdot}$$
$$n \cdot 1$$

Therefore,

$$\cdot \, f \cdot \; \cdot \cdot \; \sin m \cdot \; d \cdot \; \cdot \; b_m \cdot$$

$$b_m \cdot \frac{1}{\cdot} \cdot \, f \cdot \; \cdot \; \cdot \; \sin m \cdot \; d \cdot \cdot \; m \; 1, 2, \cdot$$

The coefficients are:

$$a_0 \cdot \frac{1}{2\cdot} \bullet_{\cdot\;\cdot}^{\cdot} f\cdot \;\cdot\; \cdot \; d\cdot$$

$$a_m \cdot \frac{1}{\cdot} \bullet_{\cdot\;\cdot}^{\cdot} f\cdot \;\cdot\; \cdot \; \cos m\cdot \; d\cdot \;\; m\cdot \; 1, 2, \cdot$$

$$b_m \cdot \frac{1}{\cdot} \bullet_{\cdot\;\cdot}^{\cdot} f\cdot \;\cdot\; \cdot \; \sin m\cdot \; d\cdot \;\; m\cdot \; 1, 2, \cdot$$

We can write *n* in place of *m*:

$$a_0 \cdot \frac{1}{2\cdot} \bullet f\cdot \; \cdot \; \cdot \; d\cdot$$

$$a_n \cdot \frac{1}{\cdot} \bullet \; f\cdot \; \cdot \; \cdot \; \cos n\cdot \; dn\cdot \; 1,2,\cdot$$

$$b_n \cdot \frac{1}{\cdot} \bullet \; f\cdot \; \cdot \; \cdot \; \sin n\cdot \; dn\cdot \; 1,2,\cdot$$

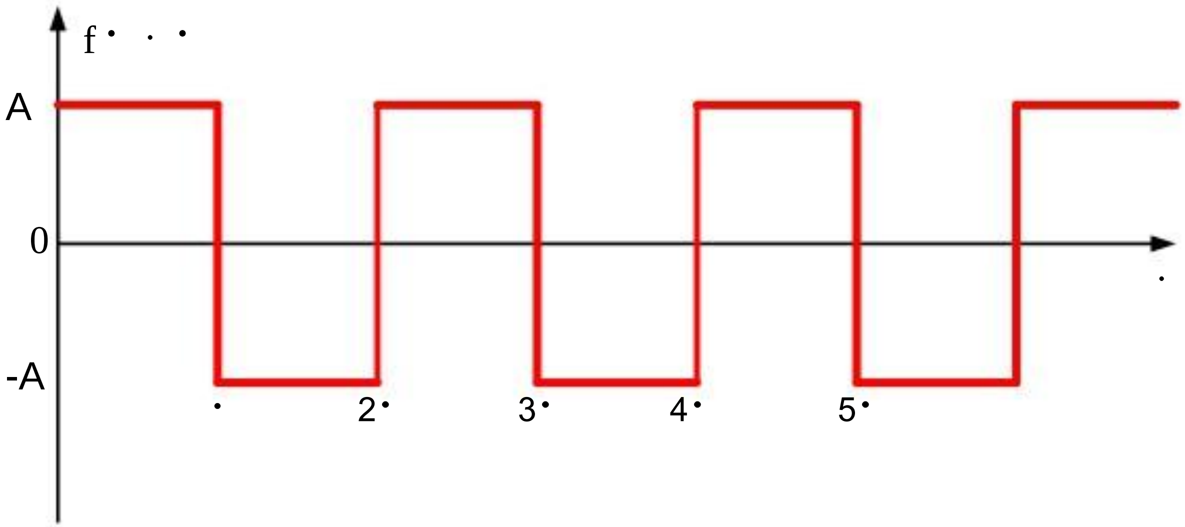
The integrations can be performed from 0 to 2π instead.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta \quad n = 1,2,\dots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta \quad n = 1,2,\dots$$

Example 1. Find the Fourier series of the following periodic function.



f· · · · *A when* 0· · · ·

· · *A when* · · · · 2·

f· · · 2· · · *f*· · ·

$$0 \cdot \frac{1}{2\cdot} \bullet \int_{0}^{2\cdot} f \cdot \cdot \; \cdot d\cdot$$

$$\cdot \; \frac{1}{2\cdot} \vdots \bullet_{0}^{\cdot} f\cdot \; \cdot \; d\cdot \; \cdot \; \bullet_{\cdot}^{2\cdot} f\cdot \; \cdot \; \cdot \; d\cdot$$

$$\cdot \; \frac{1}{2\cdot} \vdots \bullet_{0}^{\cdot} Ad\cdot \; \cdot \; \bullet_{\cdot}^{2\cdot} \; \cdot \; Ad\cdot \; \vdots$$

$$\cdot \; 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\cos n\theta \, d\theta + \int_{\pi}^{2\pi} (-A)\cos n\theta \, d\theta\right]$$

$$= \frac{1}{\pi}\left[A\frac{\sin n\theta}{n}\right]_0^{\pi} - \frac{1}{\pi}\left[A\frac{\sin n\theta}{n}\right]_{\pi}^{2\pi} = 0$$

$$\cdot \frac{1}{\cdot} \bullet_{0}^{2\cdot} f\cdot\cdot \quad \cdot \; \sin\cdot \; d\cdot$$

$$\cdot \frac{1}{\cdot} \bullet_{0}^{\cdot} A \sin n\cdot \; d\cdot \quad \cdot \bullet^{2}_{\cdot} \cdot\cdot \; \cdot \; A\cdot \; \sin n\cdot \; d\cdot$$

$$\cdot \frac{1}{\cdot} \cdot \; \cdot A \frac{\cos n\cdot}{n} \Big|_{0}^{\cdot} \cdot \frac{1}{\cdot} \cdot A \frac{\cos n\cdot}{n} \Big|_{\cdot}^{2\cdot}$$

$$\cdot \frac{A}{n\cdot} \cdot \quad \cdot \; \cos n\cdot\cdot \; \cos 0\cdot \; \cos 2n\cdot \quad \cdot \; \cos n\cdot \quad \cdot$$

$$_n \cdot \frac{A}{n\cdot} \cdot \cdot \cos n \cdot \cdot \cos 0 \cdot \cos 2n \cdot \cdot \cos n \cdot \cdot$$

$$\cdot \frac{A}{n\cdot} \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot$$

$$\cdot \frac{4A}{n\cdot}$$ when n is odd

$$a_n = \frac{A}{n}\cdot\cdot\cdot \cos n\cdot\cdot\cdot \cos 0\cdot \quad \cos 2n\cdot\cdot\cdot \cos n\cdot\cdot$$

$$\cdot\ \frac{A}{n\cdot}\cdot\ \cdot\cdot 1\ 1\cdot\ 1\cdot\ 1\cdot$$

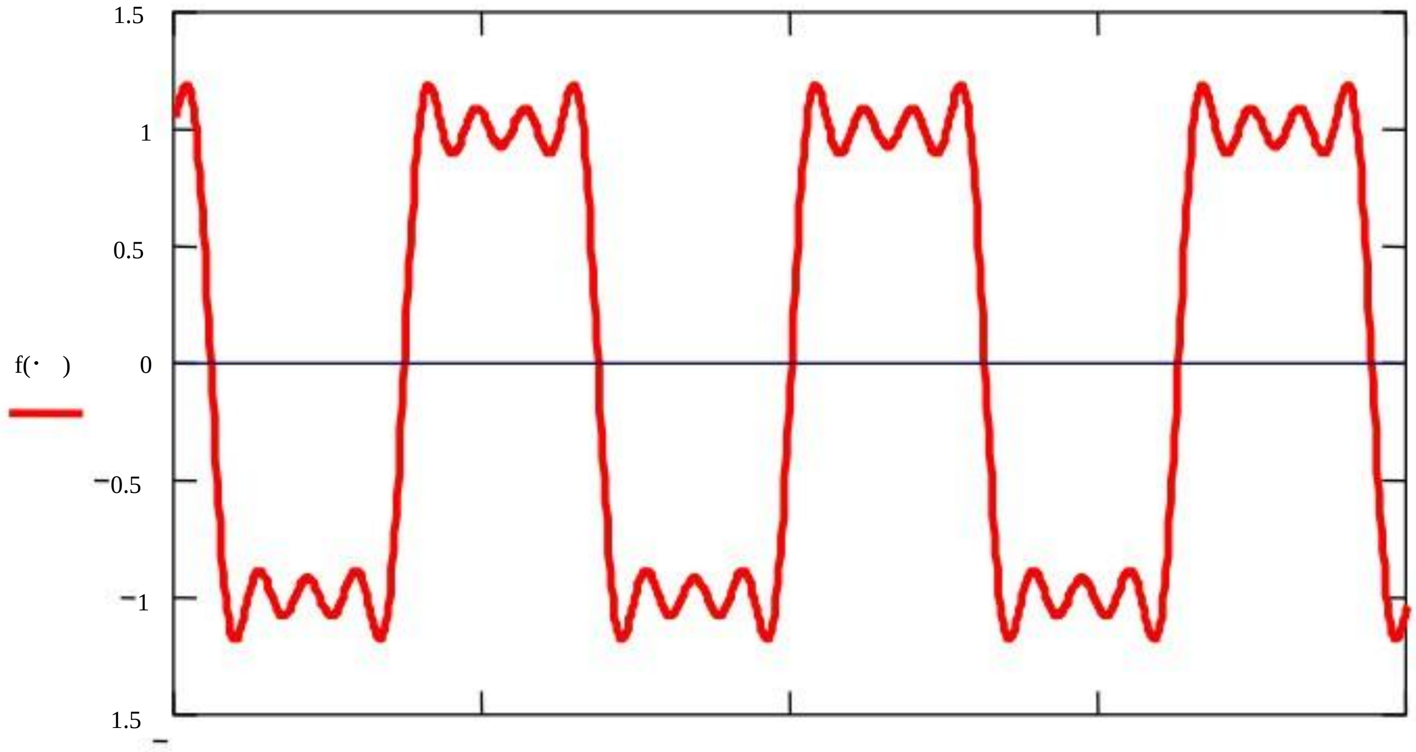
· 0 when n is even

Therefore, the corresponding Fourier series is

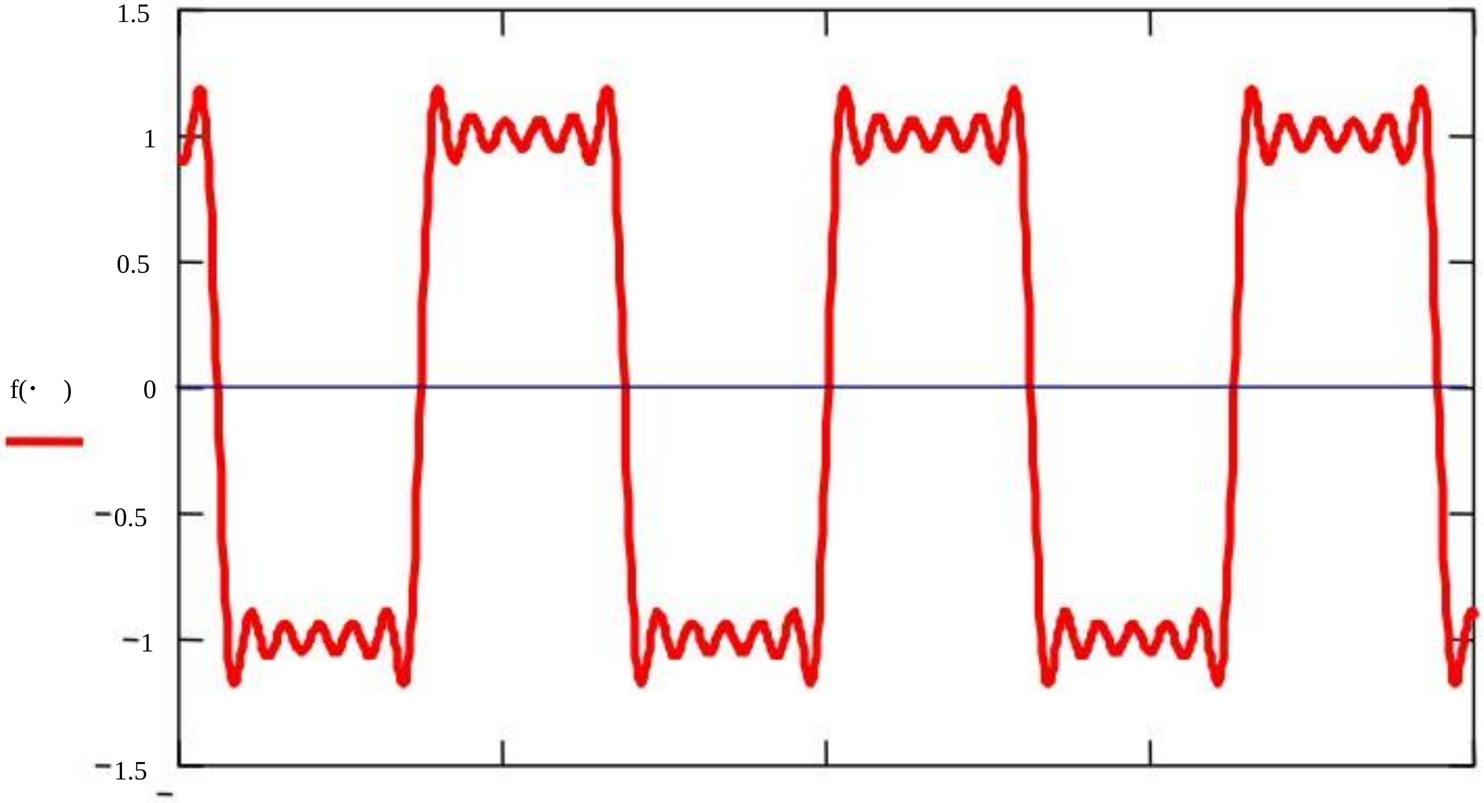
$$\frac{4A\cdot}{\cdot\quad\cdot}\cdot \sin\cdot\ \cdot\ \frac{1}{3}\sin 3\cdot\ \cdot\ \frac{1}{5}\sin 5\cdot\ \cdot\ \frac{1}{7}\sin 7\cdot\ \cdot\ \cdot\ \cdot\ \vdots$$

In writing the Fourier series we may not be able to consider infinite number of terms for practical reasons. The question therefore, is - how many terms to consider?

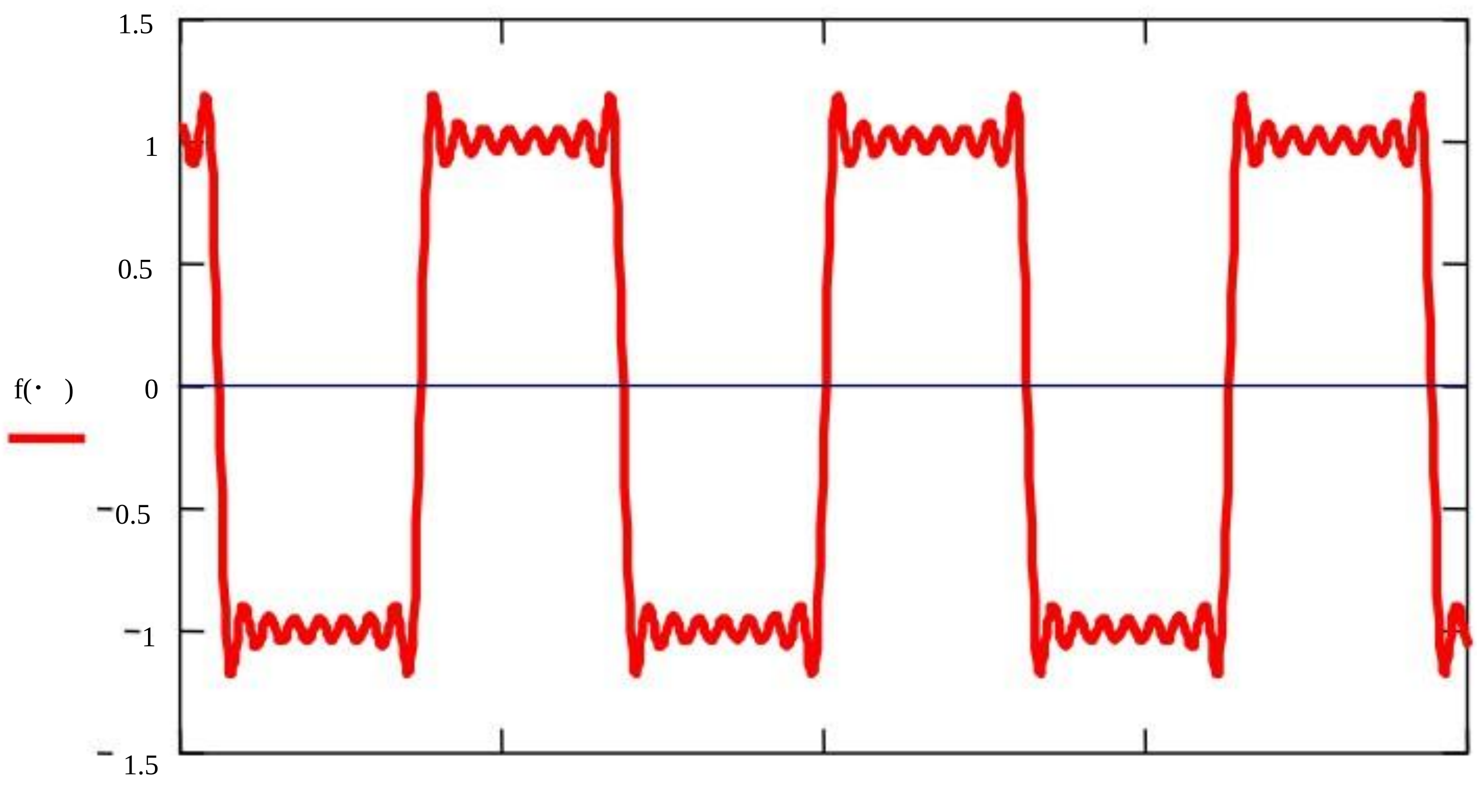
When we consider 4 terms as shown in the previous slide, the function looks like the following.



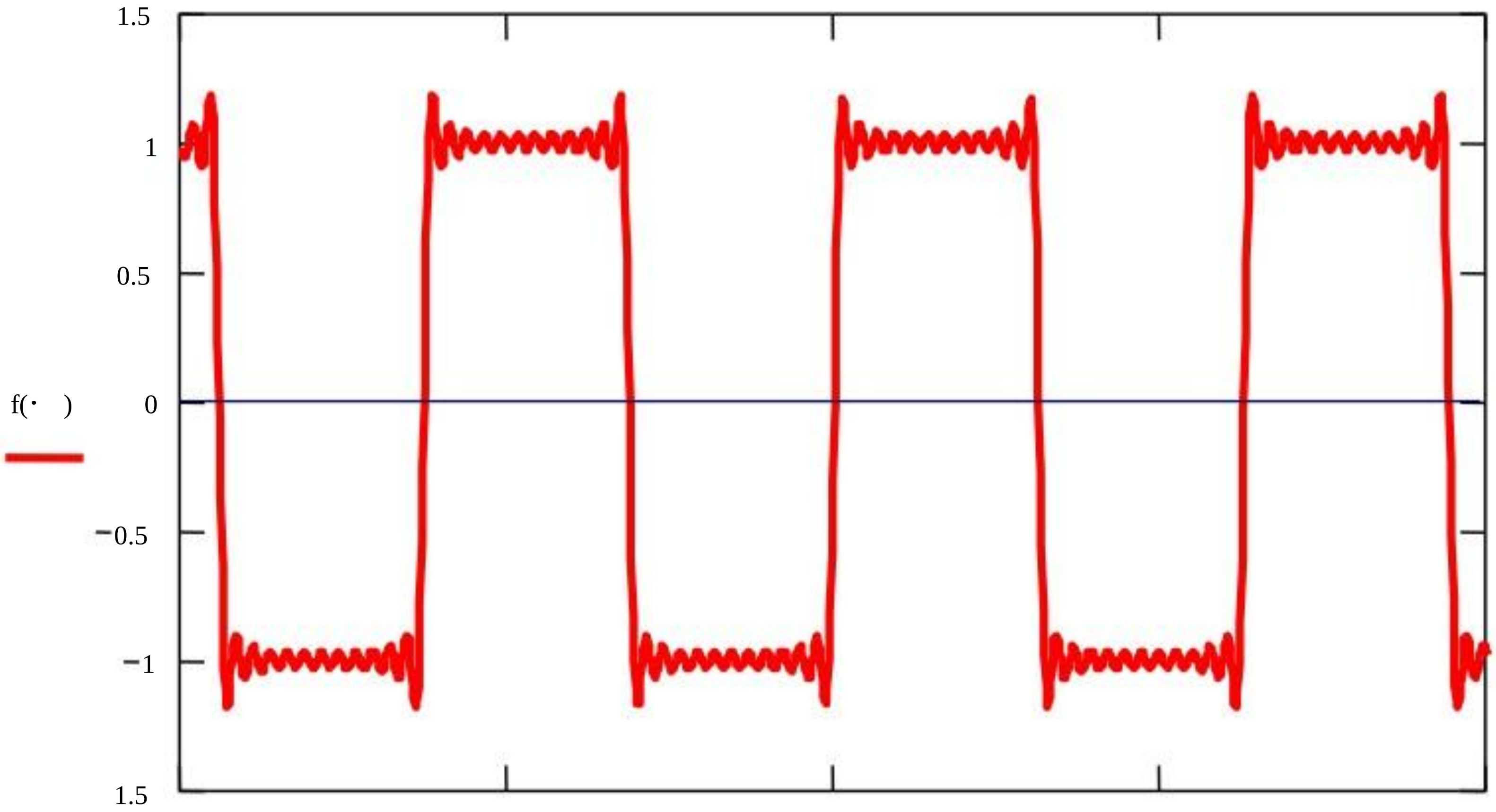
When we consider 6 terms, the function looks like the following.



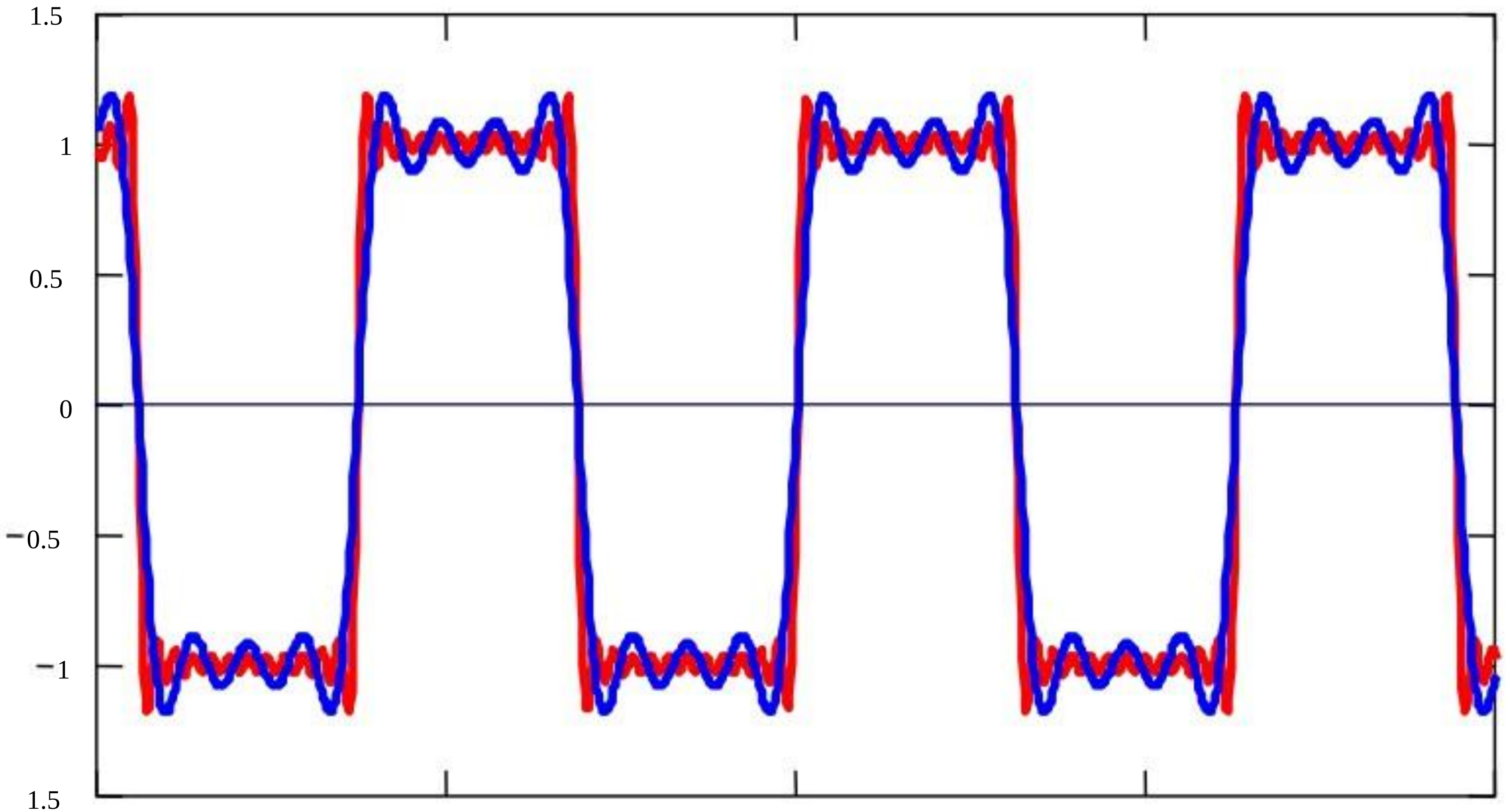
When we consider 8 terms, the function looks like the following.



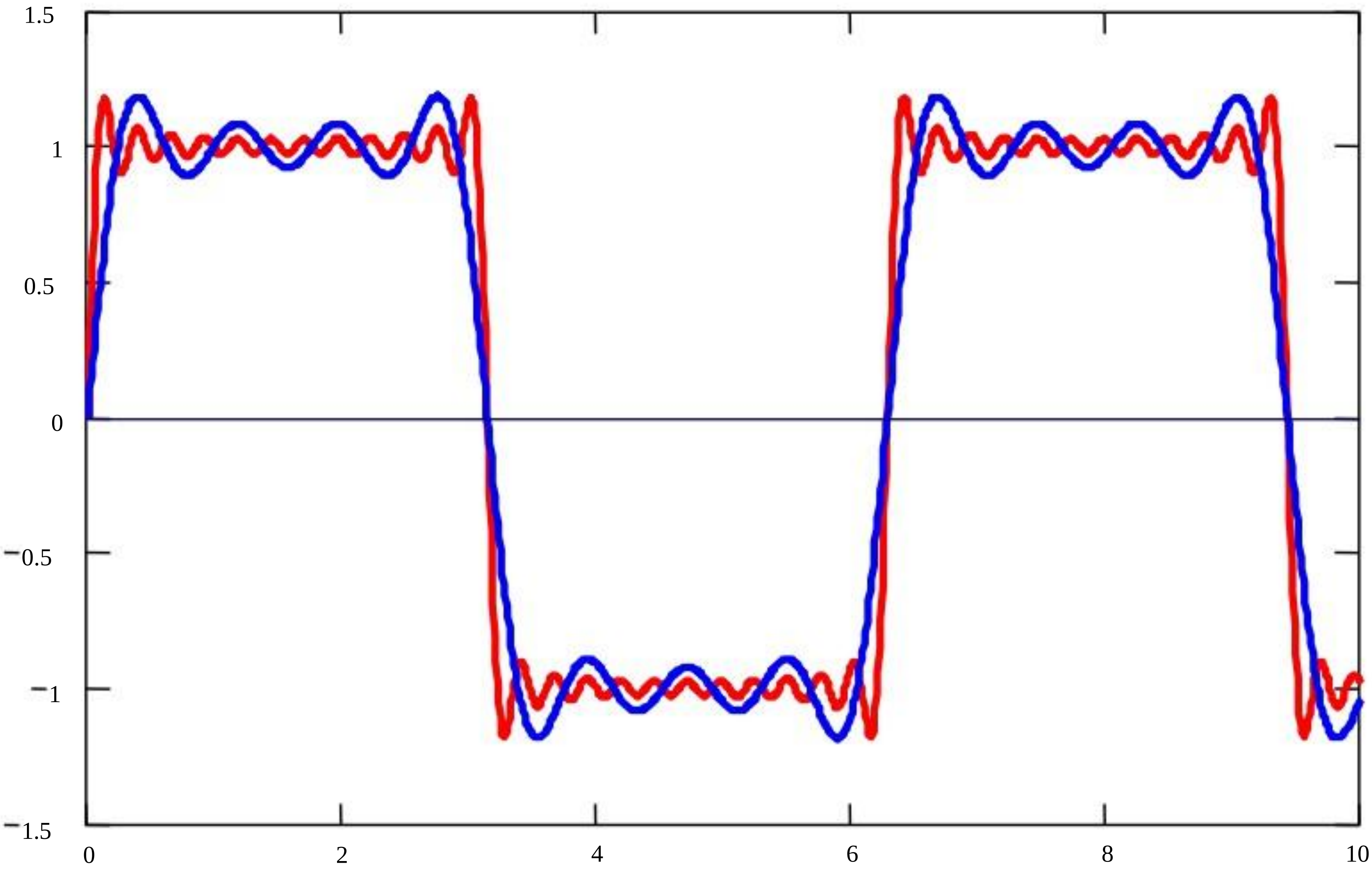
When we consider 12 terms, the function looks like the following.



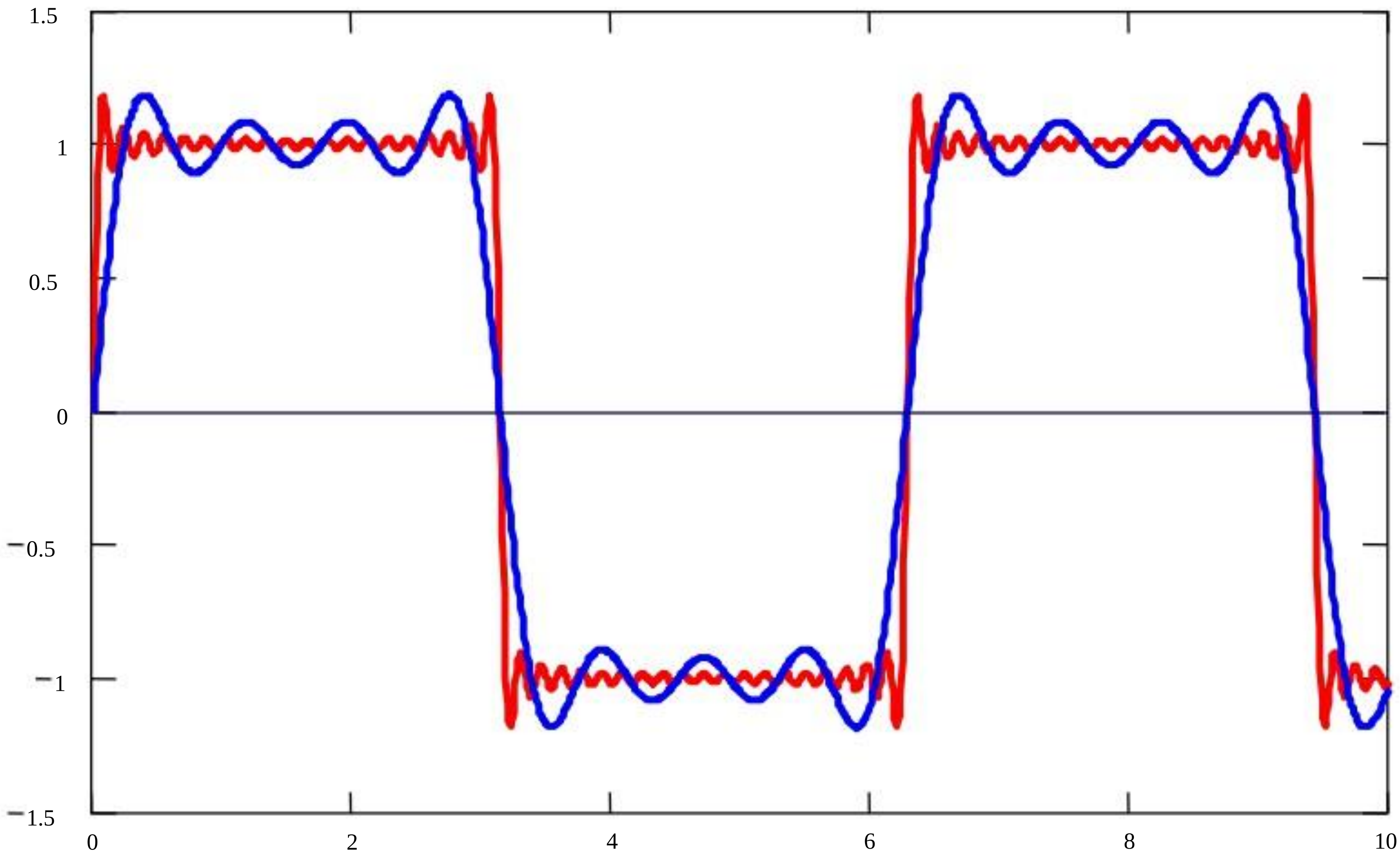
The red curve was drawn with 12 terms and the blue curve was drawn with 4 terms.



The red curve was drawn with 12 terms and the blue curve was drawn with 4 terms.



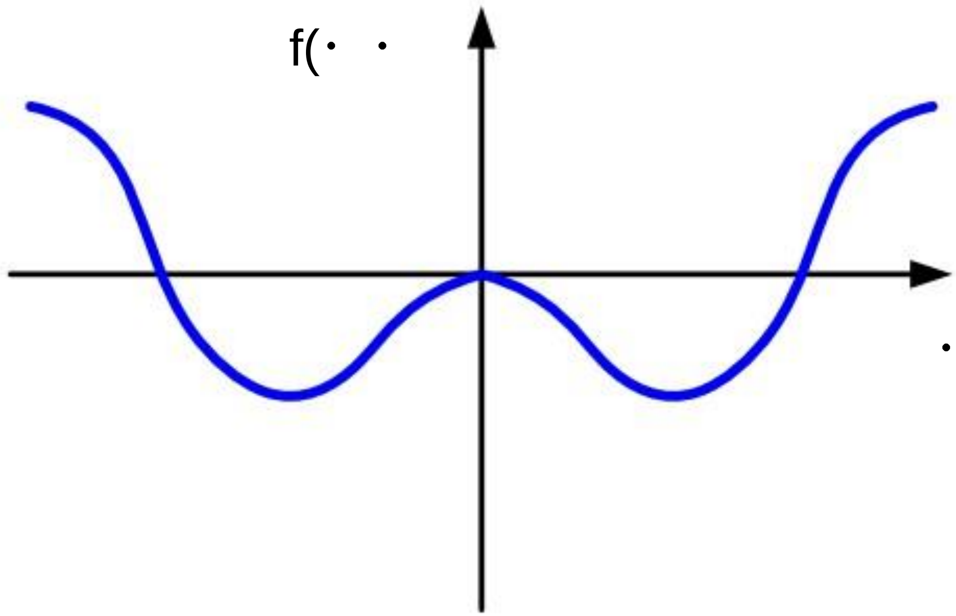
The red curve was drawn with 20 terms and the blue curve was drawn with 4 terms.



Even and Odd Functions

(We are not talking about even or odd numbers.)

Even Functions

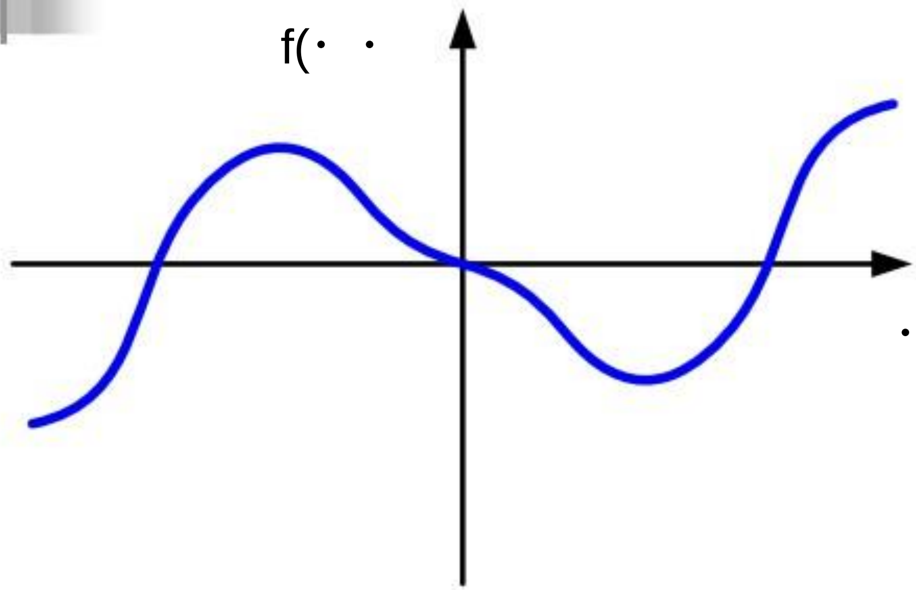


The value of the function would be the same when we walk equal distances along the X-axis in opposite directions.

Mathematically speaking -

f • • • • • *f* • • •

Odd Functions

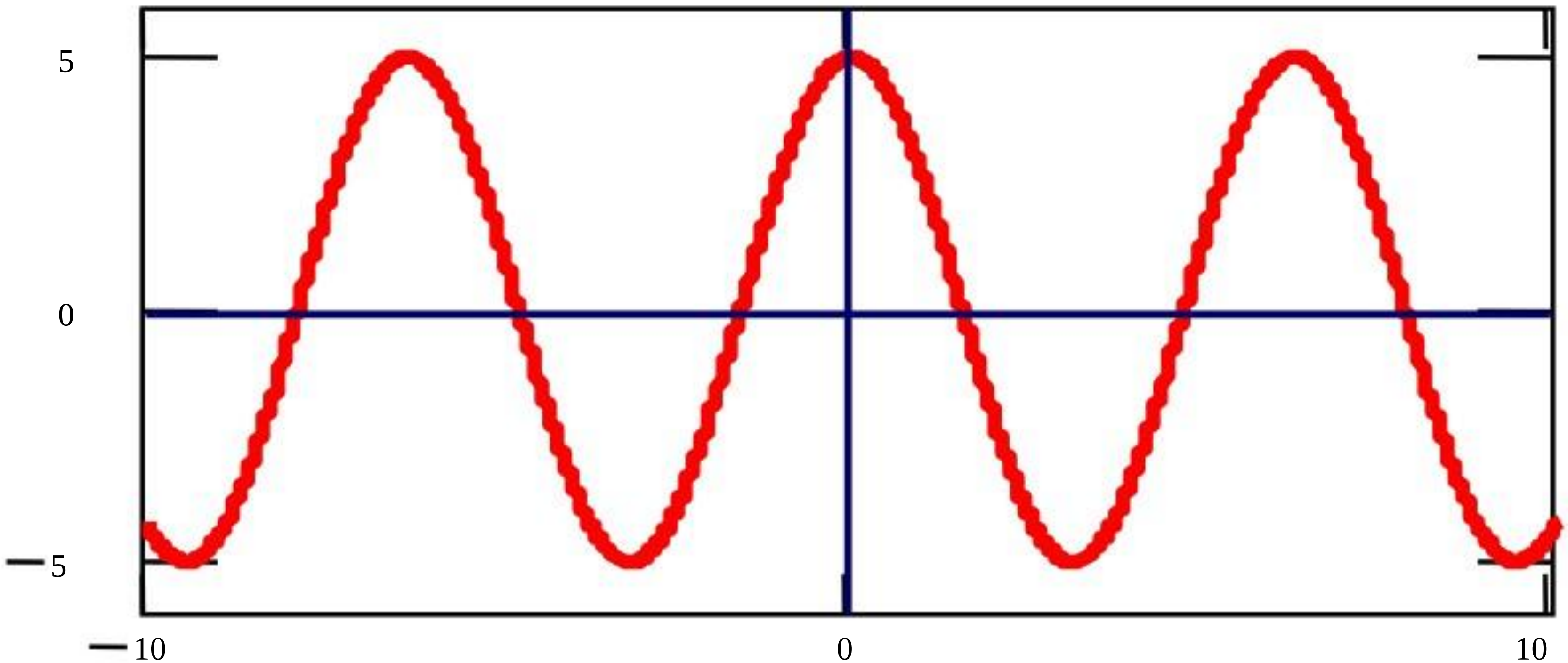


The value of the function would change its sign but with the same magnitude when we walk equal distances along the X-axis in opposite directions.

Mathematically speaking -

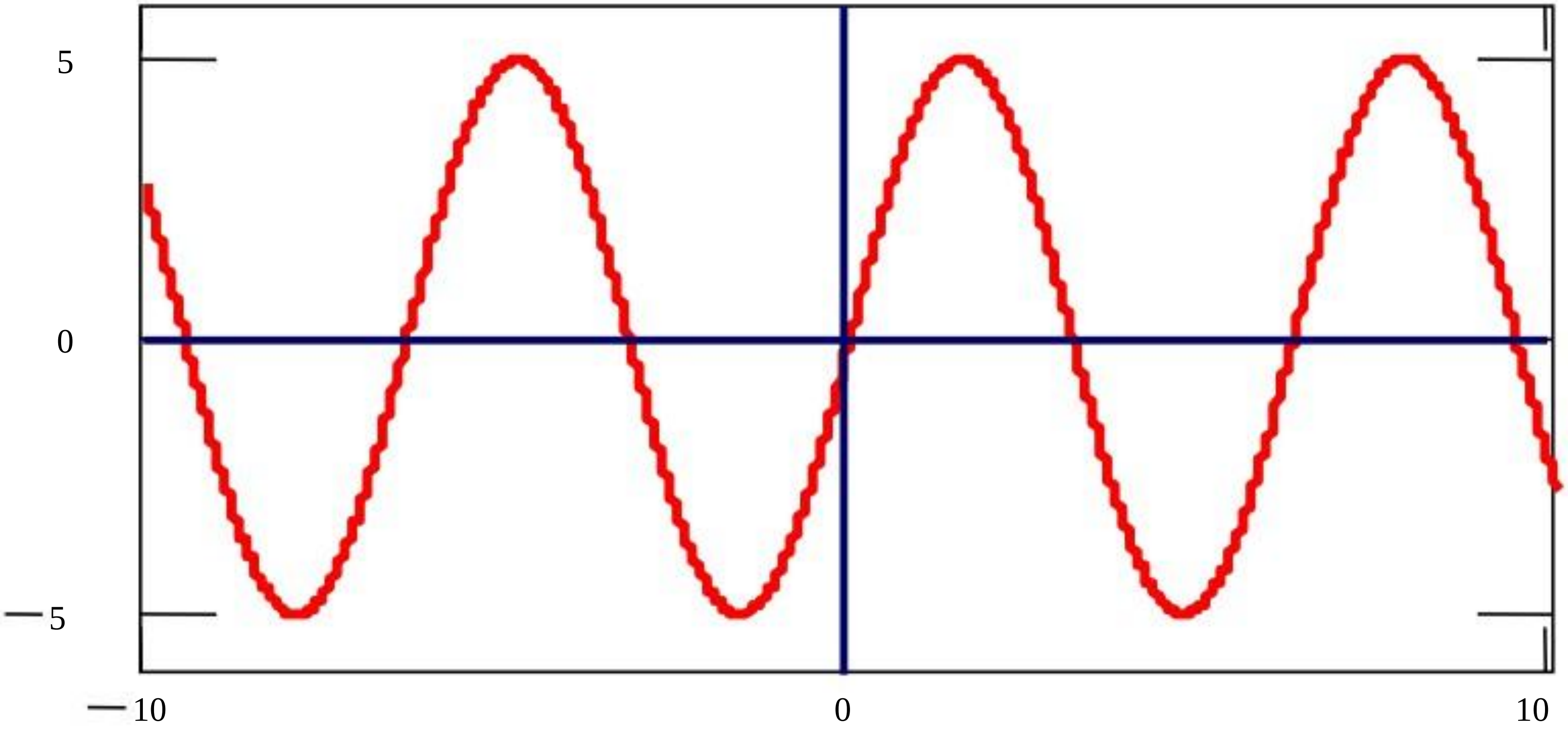
f · · · · · · *f* · · ·

Even functions can solely be represented by cosine waves because, cosine waves are even functions. A sum of even functions is another even function.



.

Odd functions can solely be represented by sine waves because, sine waves are odd functions. A sum of odd functions is another odd function.



.

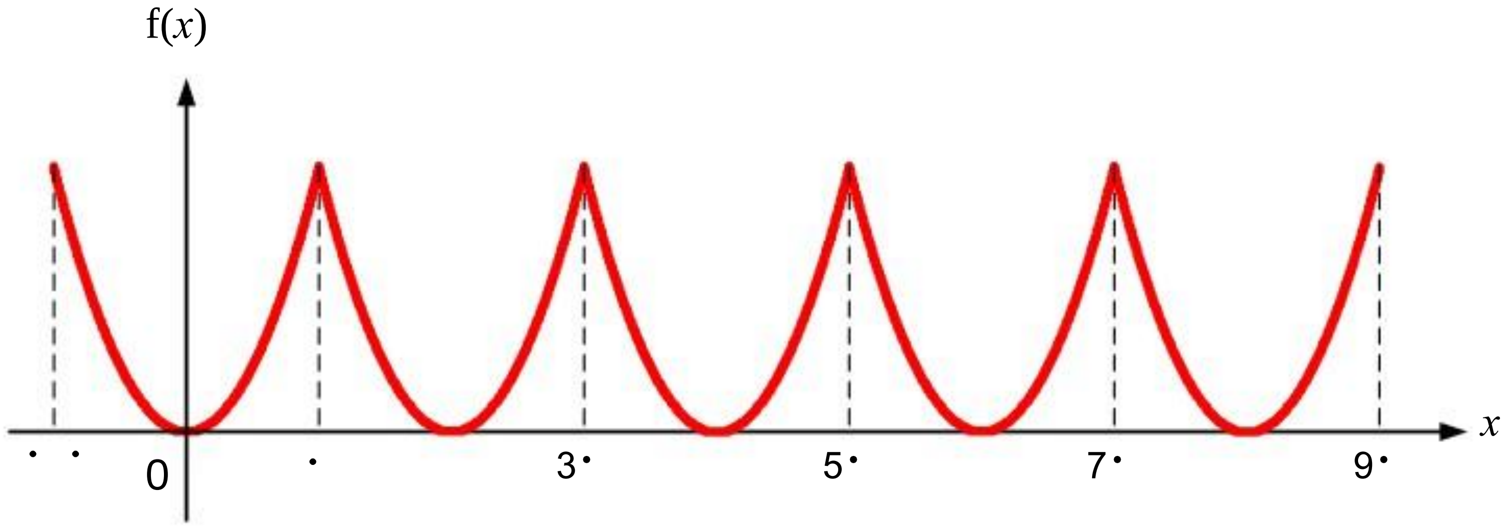
The Fourier series of an even function $f(x)$ is expressed in terms of a cosine series.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

The Fourier series of an odd function $f(x)$ is expressed in terms of a sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Example 2. Find the Fourier series of the following periodic function.



***f* • *x* • • x^2 *when* • • *x* • •**

***f* • • 2 • • • *f* • • •**

$$a_0 \cdot \frac{1}{2\cdot} \cdot \int_{\cdot\,\cdot}^{\cdot} f\,\cdot x \cdot\, dx \cdot \frac{1}{2\cdot} \cdot \int_{\cdot\,\cdot}^{\cdot} x^2\,dx$$

$$\cdot\ \frac{1}{2\cdot} \cdot \frac{x^3}{3} \cdot \Big|_{x\cdot\ \cdot\ \cdot}^{x\cdot\ \cdot} \cdot\ \frac{\cdot\ 2}{3}$$

$$\cdot \quad \frac{1}{\cdot} \; \cdot \; f \cdot x \cdot \; \cos nx\,dx$$

$$\cdot \quad \frac{1}{\cdot} \; \cdot \; x^{2} \cos nx\,dx \; \cdot$$

Use integration by parts. Details are shown in your class note.

$$\cdot \; \frac{4}{n^2}\cos n \cdot$$

$$a_n \cdot \; \cdot \; \frac{4}{n^2} \quad \text{whennisodd}$$

$$a_n \cdot \; \frac{4}{n^2} \quad \text{whenniseven}$$

This is an even function.

Therefore, $b_n \cdot 0$

The corresponding Fourier series is

$$\frac{\cdot^{2}}{3} \cdot 4 \cdot \cos x \cdot \frac{\cos 2x}{2^{2}} \cdot \frac{\cos 3x}{3^{2}} \cdot \frac{\cos 4x}{4^{2}} \cdot \cdot \cdot$$

Functions Having Arbitrary Period

Assume that a function $f\cdot \ t\cdot$ has period, T. We can relate angle (·) with time (t) in the following manner.

· · w t

w **is the angular velocity in radians per second.**

$$\cdot \quad 2\cdot f$$

f **is the frequency of the periodic function,**

t

$$\cdot \quad \cdot \quad 2\cdot \, ft$$ **where** $$f \cdot \frac{1}{T}$$

Therefore, $$\cdot \quad \cdot \quad \frac{2\cdot}{T} t$$

$$\cdot \quad \frac{2\cdot}{T} t \qquad\qquad d\cdot \quad \cdot \quad \frac{2\cdot}{T} dt$$

Now change the limits of integration.

$$\cdot \quad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \quad \frac{2\cdot}{T} t \qquad t\cdot \quad \cdot \quad \frac{T}{2}$$

$$\cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \frac{2\cdot}{T} t \qquad t\cdot \quad \frac{T}{2}$$

$0 \cdot \frac{1}{2\cdot} \cdot f \cdot \cdot \cdot d \cdot$

$a_0 \cdot \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} t$

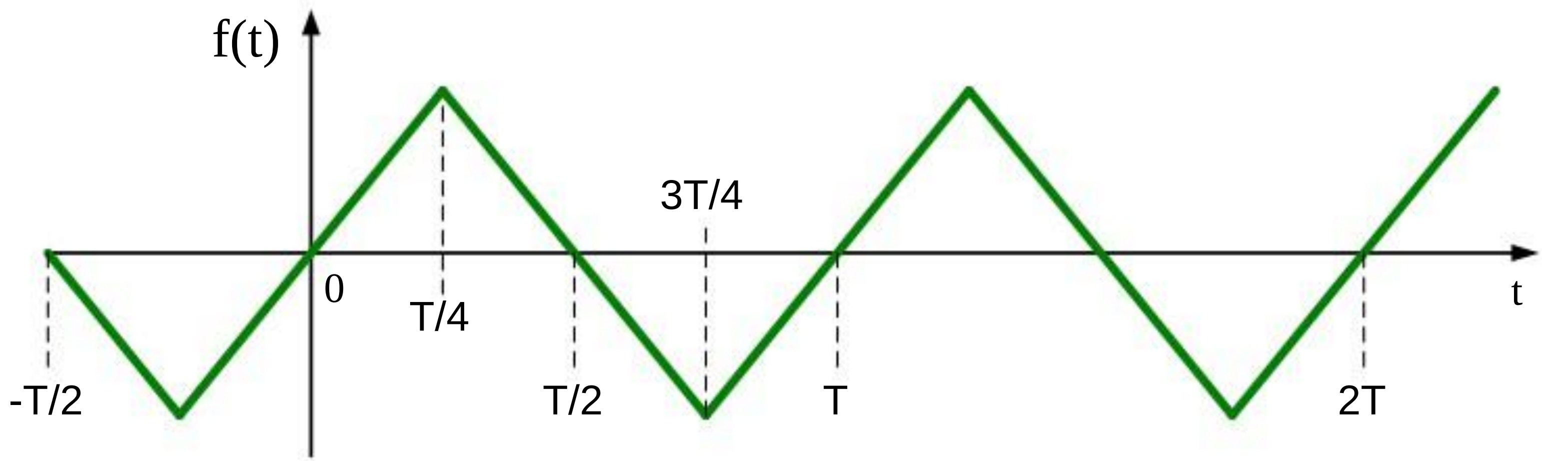
$$ {}_{n} \cdot \frac{1}{\cdot} \cdot_{\cdot\,\cdot}^{\cdot} f\cdot \cdot \cdot \cos n\cdot\; d\cdot\, n\cdot\; 1,2,\cdot $$

$$ a_n \cdot \frac{2}{T} \cdot_{\cdot\frac{T}{2}}^{\frac{T}{2}} f\,t\cdot \;\cos\frac{2\cdot\; n}{T}\, t\cdot \; dt \qquad n \cdot 1,2,\cdot $$

$$ {}_{n} \cdot \frac{1}{\cdot} \cdot \int_{\cdot}^{\cdot} f \cdot \cdot \cdot \sin n \cdot \; d \cdot \; n \cdot 1,2,\cdot $$

$$ b_n \cdot \frac{2}{T} \cdot \int_{\cdot \frac{T}{2}}^{\frac{T}{}} f t \cdot \sin \cdot \frac{2 \cdot n}{T} t \cdot \; dt \qquad n \cdot 1, 2, \cdot $$

Example 4. Find the Fourier series of the following periodic function.



$$f\,t\cdot \quad \cdot \quad t \qquad when\cdot \quad \frac{T}{4} \cdot t\cdot \quad \frac{T}{4}$$

$$\cdot \quad \cdot \quad t\cdot \quad \frac{T}{2} \qquad when \qquad \frac{T}{4} \cdot t\cdot \quad \frac{3T}{4}$$

$f(t)$

This is an odd function. Therefore, $a_n = 0$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n}{T}t\right)dt$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin\left(\frac{2\pi n}{T}t\right)dt$$

$$n \cdot \frac{4}{T} \bullet_{0}^{\frac{T}{4}} t \sin \cdot \cdot \frac{2 \cdot n}{T} t \cdot dt \cdot$$

$$\cdot \frac{4}{T} \bullet_{\frac{T}{4}}^{\frac{T}{2}} \cdot \cdot \cdot t \cdot \frac{T}{2} \cdot \cdot \sin \cdot \cdot \frac{2 \cdot n}{T} t \cdot dt \cdot$$

Use integration by parts.

$$ {}_{n} \cdot \frac{4}{T} \cdot 2 \cdot \frac{T}{2 \cdot n} \cdot^{2} \sin \cdot \frac{n \cdot}{2} \cdot \cdot $$

$$ \cdot \frac{2T}{n^{2} \cdot^{2}} \sin \cdot \frac{n \cdot}{2} \cdot $$

$b_n \cdot 0$ when *n* is even.

Therefore, the Fourier series is

$$\frac{2T\cdot}{\cdot\,^{2}}\cdot \sin\cdot\frac{2\cdot}{T}t\cdot\;\cdot\;\frac{1}{3^{2}}\sin\cdot\frac{6\cdot}{T}t\cdot\;\cdot\;\frac{1}{5^{2}}\sin\cdot\frac{10\cdot}{T}t\cdot\;\cdot\;\cdot\;\cdot$$

The Complex Form of Fourier Series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Let us utilize the Euler formulae.

The nth harmonic component of (1) can be expressed as:

$$a_n \cos n\cdot \quad \cdot \quad b_n \sin n\cdot$$

$$\cdot \quad a_n \frac{e^{jn\cdot} \quad \cdot \quad e^{\cdot jn\cdot}}{2} \cdot \quad b_n \frac{e^{jn\cdot} \quad \cdot \quad e^{\cdot jn\cdot}}{2i}$$

$$\cdot \quad a_n \frac{e^{jn\cdot} \quad \cdot \quad e^{\cdot jn\cdot}}{2} \cdot \quad ib_n \frac{e^{jn\cdot} \quad \cdot \quad e^{\cdot jn\cdot}}{2}$$

$$\cos n \cdot \quad \cdot \quad b_n \sin n \cdot$$

$$\cdot \quad \cdot \frac{a_n \cdot jb_n}{2} \cdot e^{jn\cdot} \quad \cdot \quad \cdot \frac{a_n \cdot jb_n}{2} \cdot e^{\cdot jn\cdot}$$

Denoting

$$c_n \cdot \frac{a_n \cdot jb_n}{2}, \quad c_{\cdot n} \cdot \frac{a_n \cdot jb_n}{2}$$

and $c_0 \cdot a_0$

$$a_n \cos n\cdot \quad \cdot \quad b_n \sin n\cdot$$

$$\cdot \quad c_n e^{jn\cdot} \quad \cdot \quad c_{\cdot n} e^{\cdot jn\cdot}$$

The Fourier series for $f\bullet \cdot \bullet$ can be expressed as:

$$f\bullet \cdot \bullet c_0 \cdot \sum_{n\cdot 1}^{\cdot} \bullet c_n e^{jn\cdot} \bullet c_{\cdot n} e^{\cdot jn\cdot} \bullet$$

$$\cdot \sum_{n\cdot \cdot \cdot}^{\cdot} c_n e^{jn\cdot}$$

The coefficients can be evaluated in the following manner.

$$c_n = \left(\frac{a_n - j b_n}{2} \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta - \frac{j}{2\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \left(\cos n\theta - j \sin n\theta \right) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, e^{-jn\theta} \, d\theta$$

$$\cdot \; \frac{q_n \cdot jb_n}{2}$$

$$\cdot \; \frac{1}{2\cdot} \cdot f\cdot \cdot \cos n\cdot \; d\cdot \cdot \frac{j}{2\cdot} \cdot f\cdot \cdot \cdot \sin n\cdot \; d\cdot$$

$$\cdot \; \frac{1}{2\cdot} \cdot f\cdot \cdot \cdot \cos n\cdot \cdot j\sin n\cdot \cdot d\cdot$$

$$\cdot \; \frac{1}{2\cdot} \cdot f\cdot \cdot \cdot e_{jn} d\cdot$$

$$_{n} \cdot \quad \cdot \frac{a_n \cdot jb_n}{2} \cdot \qquad c_{\cdot n} \cdot \cdot \frac{a_n \cdot jb_n}{2} \cdot$$

Note that $c_{\cdot n}$ is the complex conjugate of c_n. Hence we may write that

$$c_n \cdot \frac{1}{2\cdot} \bullet f \cdot \cdot\cdot e^{\cdot jn\cdot} d\cdot$$

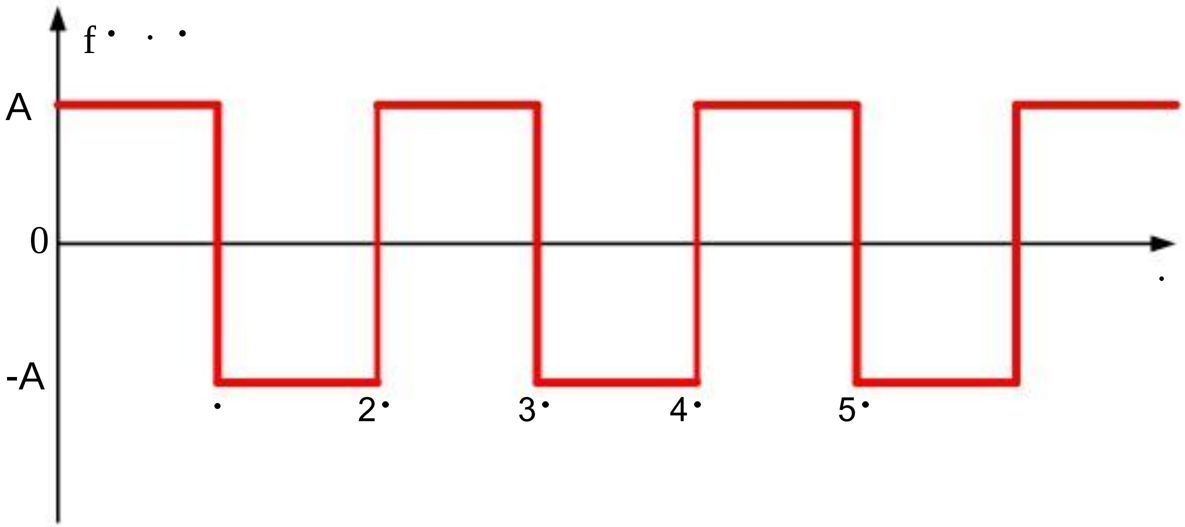
$$n \cdot \; 0, \cdot \; 1, \cdot \; 2, \cdot$$

The complex form of the Fourier series of

$f\cdot \; \cdot \; \cdot$ **withperiod** $2 \cdot$ **is:**

$$f\cdot \quad \cdot \quad \cdot \quad \cdot\cdot \quad \overset{\cdot}{\underset{n\cdot \; \cdot \; \cdot}{}} \quad c_n e^{jn\cdot}$$

Example 1. Find the Fourier series of the following periodic function.



f· · · · *A when* 0· · · ·

· · *A when* · · · · 2·

f· · · 2· · · *f*· · ·

5

f(x) · A if 0 · x · ·
· A if · · x · 2 · ·
0 otherwise

A0 · 1 / 2· · · · 2· · f(x)dx 0

A0 · 0

1 · · 8

$$A_n \cdot \frac{1}{\cdot \quad \cdot} \cdot \int_{0}^{2\cdot} f(x) \cdot \cos(n \cdot x)dx$$

$A_1 \cdot 0$ $A_2 \cdot 0$ $A_3 \cdot 0$ $A_4 \cdot 0$

$A_5 \cdot 0$ $A_6 \cdot 0$ $A_7 \cdot 0$ $A_8 \cdot 0$

$$B_n \cdot \frac{1}{\cdot} \cdot \cdot \int_{0}^{2\cdot} f(x) \cdot \sin(n \cdot x)\,dx$$

$B_1 \cdot 6.366$	$B_2 \cdot 0$	$B_3 \cdot 2.122$	$B_4 \cdot 0$
$B_5 \cdot 1.273$	$B_6 \cdot 0$	$B_7 \cdot 0.909$	$B_8 \cdot 0$

Complex Form

$$f\cdot \; \cdot \; \cdot \; \bullet \quad c_n e^{jn\cdot}$$

$$n\cdot \; \cdot \; \cdot$$

$$c_n \cdot \frac{1}{2\cdot} \bullet f\cdot \; \cdot \; \cdot \; e^{\cdot jn\cdot} d\cdot$$

$$n\cdot \; 0, \cdot \; 1, \cdot \quad ,$$

$$C(n) \quad \cdot \quad \frac{1}{2\cdot} \cdot \; \cdot \; \overset{2\cdot}{\underset{0}{}} \; f(x)\cdot \; e\cdot \; 1i\cdot \; n\cdot \; x \, dx$$

$$C(n) \cdot \frac{1}{2\cdot} \cdot \overset{2\cdot}{\underset{0}{\cdot}} f(x)\cdot \; e^{\cdot 1i\cdot n\cdot x}\, dx$$

$(\;)\cdot \; 0$ | $C(1) \cdot \; \cdot \; 3.183i$ | $C(2) \cdot \; 0$ | $C(3) \cdot \; \cdot \; 1.061i$

$C(4) \cdot \; 0$ | $C(5) \cdot \; \cdot \; 0.637i$ | $C(6) \cdot \; 0$ | $C(7) \cdot \; \cdot \; 0.455i$

$C(\cdot \; 1) \cdot \; 3.183i$ | $C(\cdot \; 2) \cdot \; 0$ | $C(\cdot \; 3) \cdot \; 1.061i$

$C(\cdot \; 4) \cdot \; 0$ | $C(\cdot \; 5) \cdot \; 0.637i$ | $C(\cdot \; 6) \cdot \; 0$ | $C(\cdot \; 7) \cdot \; 0.455i$

The Fourier Series

Recall from calculus that sinusoids whose frequencies are integer multiples of some fundamental frequency $f_0 = 1/T$ form an ***orthogonal*** set of functions.

$$\frac{2}{T}\int_0^T \sin\frac{2\pi nt}{T}\cos\frac{2\pi mt}{T}\,dt = 0; \quad \forall\, n,m$$

and

$$\frac{2}{T}\int_0^T \sin\frac{2\pi nt}{T}\sin\frac{2\pi mt}{T}\,dt = \frac{2}{T}\int_0^T \cos\frac{2\pi nt}{T}\cos\frac{2\pi mt}{T}\,dt = \begin{cases} 0 & ; n \neq m \\ 1 & ; n = m \neq 0 \end{cases}$$

The Fourier Series

The Fourier Trigonometric Coefficients can be obtained from

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

average value over one period

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\,dt \qquad n > 0$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\,dt \qquad n > 0$$

The Fourier Series

To obtain a_k

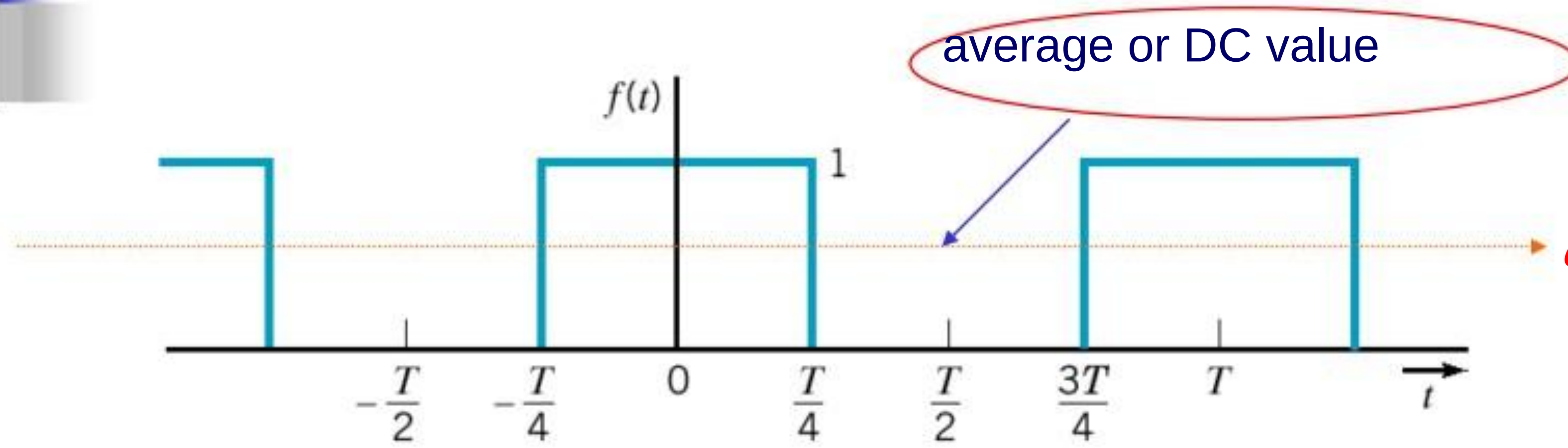
$$\int_0^T f(t)\cos k\omega_0 t\,dt = \int_0^T a_0 \cos k\omega_0 t\,dt + \sum_{n=1}^{N}\int_0^T \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)\cos k\omega_0 t\,dt$$

The only nonzero term is for $n = k$

$$\int_0^T f(t)\cos k\omega_0 t\,dt = a_k\left(\frac{T}{2}\right)$$

Similar approach can be used to obtain b_k

Example 1 determine Fourier Series and plot for N = 7



$a_0 = \frac{1}{2}$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{T}\int_{-T/4}^{T/4} 1dt = \frac{1}{2}$$

Example 1(cont.)

An ***even function*** exhibits symmetry around the vertical axis at *t = 0* so that *f(t) = f(-t).*

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega t\, dt_0$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} 1 \sin n\omega t\, dt_0 = 0$$

Determine only a_n

$$a_n = \frac{2}{T}\int_{-T/4}^{T/4} 1 \cos n\omega_0 t\, dt$$

$$= \frac{2}{T\omega_0 n}\sin n\omega_0 t \Big|_{-T/4}^{T/4}$$

Example 1(cont.)

$$a_n = \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right)\right]$$

$$a_n = 0 \text{ when } n = 2, 4, 6, \ldots$$

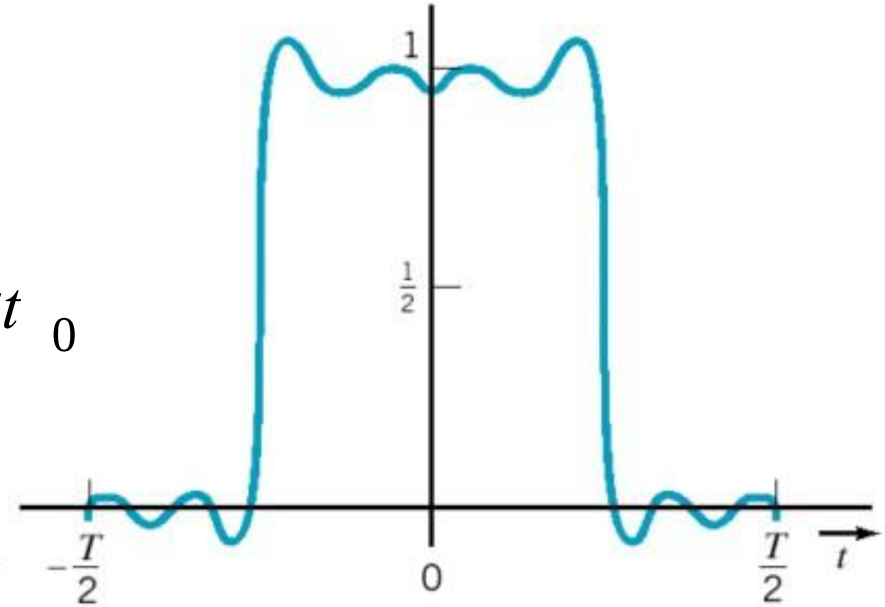
and

$$a_n = \frac{2(-1)^q}{n\pi} \text{ when } n = 1, 3, 5, \ldots$$

where $q = \dfrac{(n-1)}{2}$

$$f(t) = \frac{1}{2} + \sum_{n=1,odd}^{N} \frac{2(-1)^q}{n\pi} \cos n\omega_0 t$$

$$a_1 = \frac{2}{\pi}, a_3 = \frac{-2}{3\pi}, a_5 = \frac{2}{5\pi}, a_7 = \frac{-2}{7\pi}$$



Symmetry of the Function

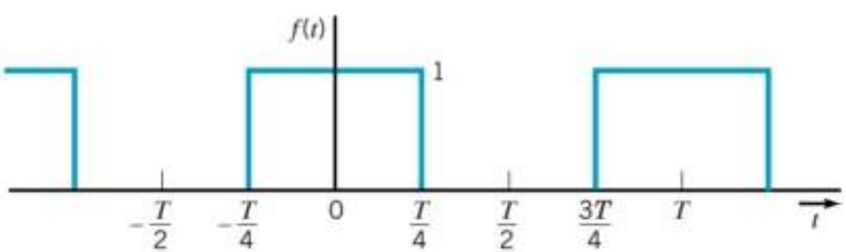
Four types

1. Even-function symmetry
2. Odd-function symmetry
3. Half-wave symmetry
4. Quarter-wave symmetry

Even function

$$f(t) \cdot \; f(\cdot \;\; t)$$

All $b_n = 0$

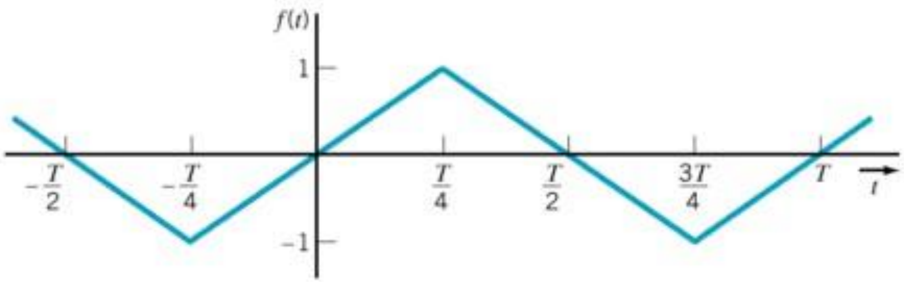


$$a_n \cdot \frac{4}{T} \bullet_0^{T/2} \; f\ t)\cos n\mathrm{w} \qquad {}_0 t\ dt$$

Symmetry of the Function

Odd function

$$f(t) = -f(-t)$$ All $a_n = 0$



$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$$

Half-wave symmetry

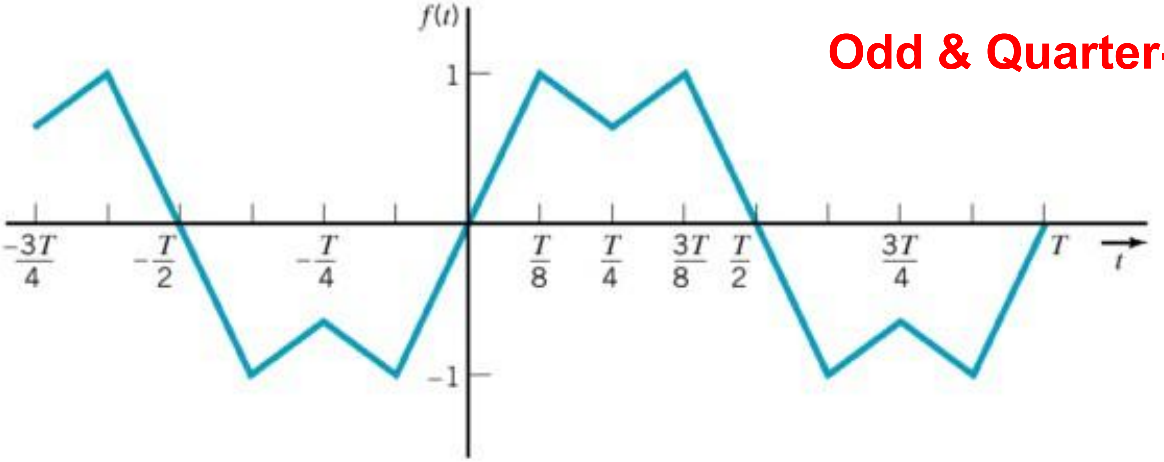
$$f(t) = -f(t - \frac{T}{2})$$

a_n and b_n = 0 for even values of n and $a_0 = 0$

Symmetry of the Function

Quarter-wave symmetry

Odd & Quarter-wave



All $a_n = 0$ and $b_n = 0$ for even values of n and $a_0 = 0$

$$b_n = \frac{8}{T}\int_0^{T/4} f(t)\sin n\omega_0 t\, dt \quad ; \text{ for odd } n$$

Symmetry of the Function

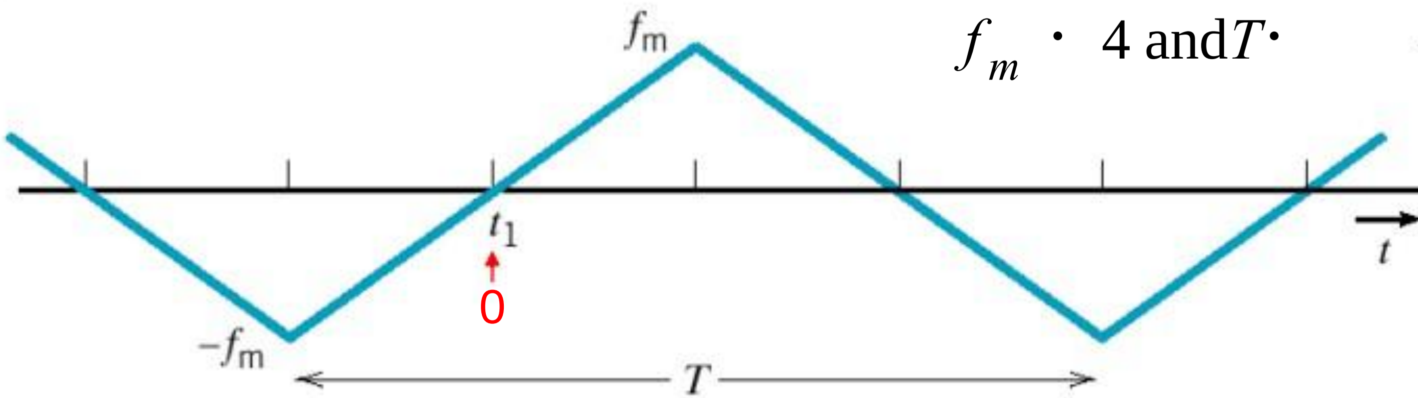
For Even & Quarter-wave

All $b_n = 0$ and $a_n = 0$ for even values of n and $a_0 = 0$

$$a_n = \frac{8}{T}\int_0^{T/4} f(t)\cos n\omega_0 t\,dt \quad ; \text{ for odd } n$$

Table 15.4-1 gives a summary of Fourier coefficients and symmetry.

Example 2 determine Fourier Series and N = ?



$$f_m = 4 \text{ and } T = \frac{\pi}{2}\ \text{s}$$

$$T = \frac{\pi}{2} \Rightarrow \omega_0 = \frac{2\pi}{T} = 4\ \text{rad/s}$$

To obtain the most advantages form of symmetry, we choose $t_1 = 0$ s

⇒ **Odd & Quarter-wave**

All $a_n = 0$ and $b_n = 0$ for even values of n and $a_0 = 0$

$$b_n = \frac{8}{T}\int_0^{T/4} f(t)\sin n\omega_0 t\,dt \quad ; \text{for odd } n$$

Example 2(cont.)

$$f(t) = \frac{f_m}{T/4}\,t = \frac{4f_m}{T}\,t \quad ; 0 \le t \le T/4$$

(annotations: $f_m = 4$, $T = \frac{\pi}{2}$)

$$\therefore f(t) = \frac{32}{\pi}\,t \quad ; 0 \le t \le T/4$$

$$b_n = \frac{8}{T}\left(\frac{32}{\pi}\right)\int_0^{T/4} t \sin n\omega t\,dt$$

$$= \frac{512}{\pi^2}\left[\frac{\sin n\omega t}{n^2\omega_0^2} - \frac{t\cos n\omega t}{n\omega_0}\right]_0^{T/4}$$

$$= \frac{32}{\pi^2 n^2}\sin\frac{n\pi}{2} \quad ; \text{for odd } n$$

Example 2(cont.)

The Fourier Series is

$$f(t) = 3.24 \sum_{n=1}^{N} \frac{1}{n^2} \sin\frac{n\pi}{2} \sin n\omega_0 t \quad ; \text{for odd } n$$

$$3.24 = \frac{32}{\pi^2}$$

The first 4 terms (upto and including N = 7)

$$f(t) = 3.24\left(\sin 4t - \frac{1}{9}\sin 12t + \frac{1}{25}\sin 20t - \frac{1}{49}\sin 28t\right)$$

Next harmonic is for N = 9 which has magnitude 3.24/81 = 0.04 < 2 % of b_1 (= 3.24)

Therefore the first 4 terms (including N = 7) is enough for the desired approximation

Exponential Form of the Fourier Series

$$f\,t) \cdot \; C_0 \cdot \bullet \sum^{N}_{n \cdot 1} C_n \cos(n \mathrm{w} t \cdot_0 \cdot\;)_n$$

C_0 is the average (or DC) value of *f(t)* and

$$\mathbf{C}_n \cdot \frac{(a_n \cdot jb)_n}{2} \cdot C\dot{}_n \cdot \;_n$$

where

$$C_n \cdot \left|\mathbf{C}_n\right| \cdot \frac{\sqrt{a_n^2 \cdot b_n^2}}{2}$$

and

$$\cdot_n \cdot \cdot \begin{cases} \tan^{\cdot 1} \cdot \dfrac{b_n}{a_n} \cdot ; \text{ if } a_n \cdot 0 \\ 180 \cdot \cdot \tan^{\cdot 1} \cdot \cdot \dfrac{b_n}{a_n} \cdot ; \text{ if } a_n \cdot 0 \end{cases}$$

Exponential Form of the Fourier Series

or

$$a_n \cdot 2C_n \cos\cdot_n \text{ and } b_n \cdot 2C_n \sin\cdot_n$$

Writing Euler"s identity with $\cos(n\omega_0 t \cdot \cdot_n)$ **n**tial form using

$$N\cdot \cdot$$

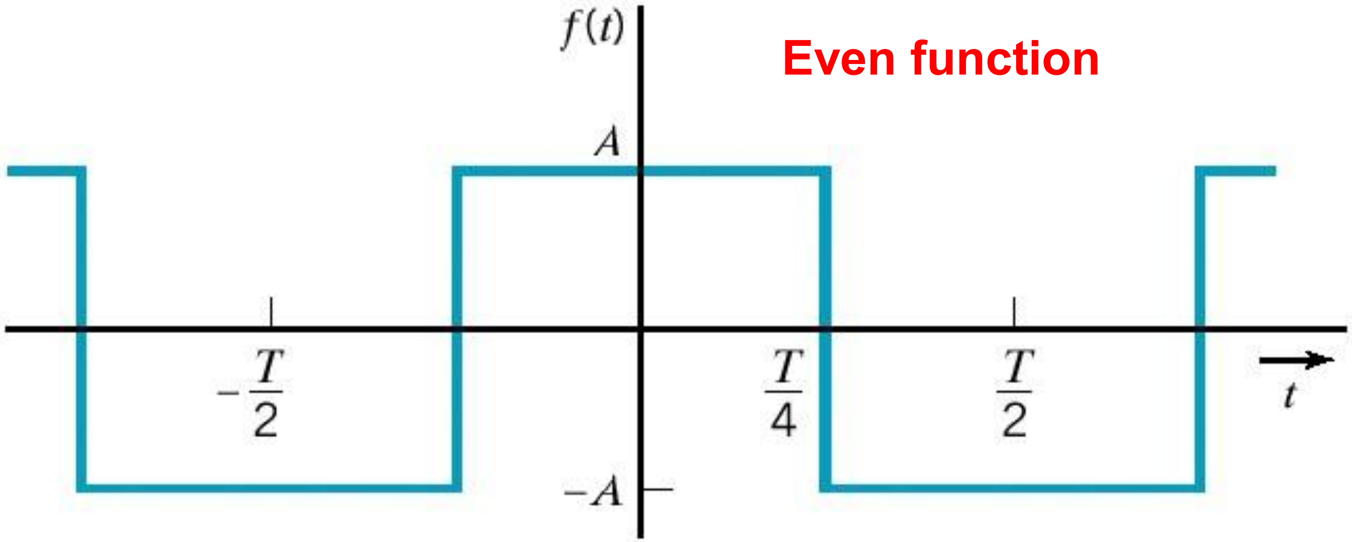
$$f(t)\cdot C_0 \cdot \sum_{n\cdot\cdot\cdot,\; n\cdot 0}^{\cdot} \mathbf{C}_n e^{jn\omega t_0} \cdot \sum_{n\cdot\cdot\cdot}^{\cdot} \mathbf{C}_n e^{jn\omega t_0}$$

where the ***complex coefficients*** are defined as

$$\mathbf{C}_n \cdot \frac{1}{T} \int_{t_0}^{t_0\cdot T} f(t) e^{\cdot jn\omega 0t}\, dt \cdot C_n e^{j\cdot_n}$$

And complex $\mathbf{C}_n$ $\mathbf{C}^*_{\cdot n}$

Example 3 determine complex Fourier Series



Even function

The average value of *f(t)* is zero $\Rightarrow C_0 = 0$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-jn\omega_0 t}\,dt$$

We select $t_0 = -\frac{T}{2}$ define $jn\omega_0 = m$

Example 3(cont.)

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{-T/4} -Ae^{-mt}dt + \frac{1}{T}\int_{-T/4}^{T/4} Ae^{-mt}dt + \frac{1}{T}\int_{T/4}^{T/2} -Ae^{-mt}dt$$

$$= \frac{A}{mT}\left[e^{-mt}\Big|_{-T/2}^{-T/4} - e^{-mt}\Big|_{-T/4}^{T/4} + e^{-mt}\Big|_{T/4}^{T/2}\right]$$

$$= \frac{A}{jn\omega_0 T}\left[2e^{jn\pi/2} - 2e^{-jn\pi/2} - e^{-jn\pi} - e^{jn\pi}\right]$$

$$= \frac{A}{2\pi n}\left[4\sin\frac{n\pi}{2} - 2\sin(n\pi)\right] = \begin{cases} 0 & ;\ \text{for even } n \\ \dfrac{2A}{n\pi}\sin\dfrac{n\pi}{2} & ;\ \text{for odd } n \end{cases}$$

$$= A\frac{\sin x}{x} \quad \text{where } x = \frac{n\pi}{2}$$

Example 3(cont.)

Since *f(t)* is even function, all C_n are real and = 0 for *n* even

For $n = 1$

$$C_1 \cdot \frac{A\sin\cdot\ /2}{\cdot\ /2} \cdot \frac{2A}{\cdot} \cdot C_{\cdot\ 1}$$

For $n = 2$

$$C_2 \cdot A\frac{\sin\cdot}{\cdot} \cdot 0 \cdot C_{\cdot\ 2}$$

For $n = 3$

$$C_3 \cdot \frac{A\sin(3\cdot\ /2)}{3\cdot\ /2} \cdot \frac{\cdot\ 2A}{3\cdot} \cdot C_{\cdot\ 3}$$

Example 3(cont.)

The complex Fourier Series is

$$f\,t) \cdot \cdot \cdot \quad \frac{\cdot\ 2A}{3\cdot} e^{\cdot\ j3\mathrm{w}_0 t} \cdot \frac{2A}{\cdot} e^{\cdot\ j\mathrm{w}_0 t} \cdot \frac{2A}{\cdot} e^{j\mathrm{w}_0 t} \cdot \frac{\cdot\ 2A}{3\cdot} e^{j3\mathrm{w}_0 t} \cdot \cdot$$

$$\cdot \frac{2A}{\cdot} \cdot e^{j\mathrm{w}_0 t} \cdot e^{\cdot\ j\mathrm{w}_0 t} \cdot \cdot \frac{\cdot\ 2A}{3\cdot} \cdot e^{j3\mathrm{w}_0 t} \cdot e^{\cdot\ j3\mathrm{w}_0 t} \cdot \cdot \cdot$$

$$\cdot \frac{4A}{\cdot} \cos \mathrm{w} t \cdot_0 \quad \frac{4A}{3\cdot} \cos 3\mathrm{w} t \cdot_0 \cdot$$

$$e^{jx} \cdot e^{\cdot\ jx} \cdot 2\cos x$$

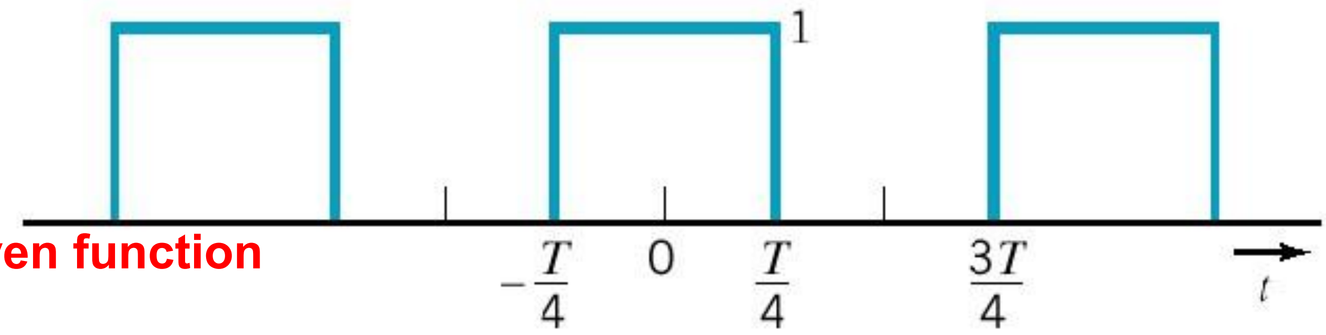
$$e^{jx} \cdot e^{\cdot\ jx} \cdot 2j\sin x$$

$$\cdot \frac{4A}{\cdot} \overset{\cdot}{\underset{\substack{n\cdot\ 1 \\ n\cdot\ odd}}{\bullet}} \frac{(\cdot\ 1)^{q}}{n} \cos n\mathrm{w} t_0$$

where $q \cdot \frac{n \cdot\ 1}{2}$

For real *f(t)* $\cdot$ $|\mathbf{C}_n| \cdot |\mathbf{C}_{\cdot n}|$

Example 4 determine complex Fourier Series



Even function

Use $jn\omega_0 \cdot m$

$$C_n \cdot \frac{1}{T} \bullet_{\cdot T/4}^{T/4} 1e^{\cdot mt}dt$$

$$\cdot \frac{1}{\cdot mT} e^{\cdot mt}\Big|_{\cdot T/4}^{T/4}$$

$$\cdot \frac{1}{\cdot mT} \bullet e^{\cdot mT/4} \bullet e^{\cdot mT/4} \bullet$$

Example 4(cont.)

$$\mathbf{C}_n \cdot \frac{1}{\cdot jn2\cdot} \cdot e^{\cdot jn\cdot /2} \cdot e^{\cdot jn\cdot /2} \cdot$$

$$\cdot\ 0 \quad ;n \quad \text{even}, n \cdot 0$$

$$\cdot\ \cdot\ \cdot\ (\cdot 1)^{(n\cdot 1)/2} \ ;\ n \text{odd}$$

To find C_0

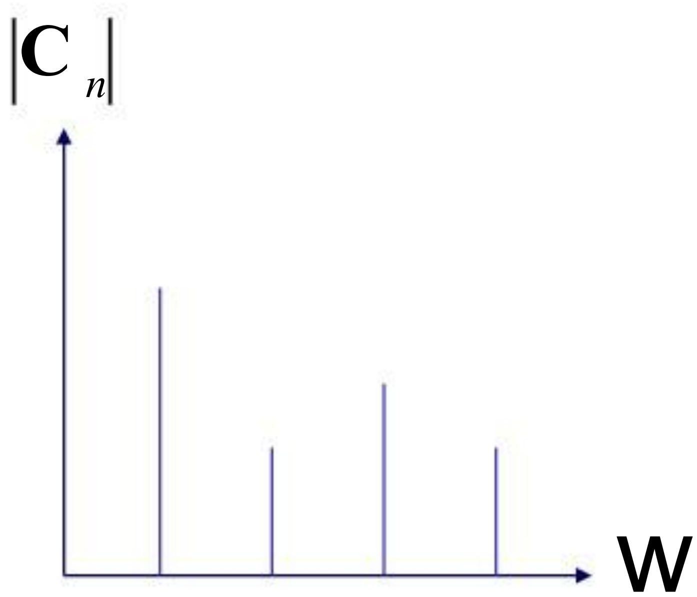
$$C_0 \cdot \frac{1}{T} \cdot_0^T f\,t)dt$$

$$\cdot \frac{1}{T} \cdot_{\cdot T/4}^{T/4} 1dt \cdot \frac{1}{2}$$

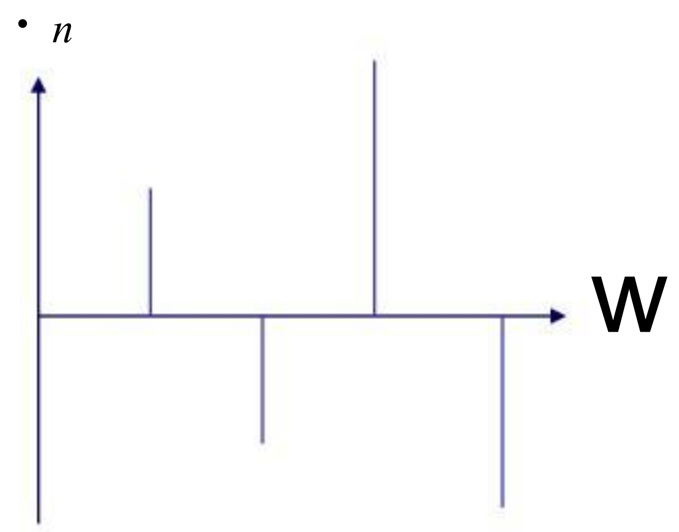
The Fourier Spectrum

The complex Fourier coefficients

$$C_n \cdot |C_n| \cdot \cdot_n$$



Amplitude spectrum

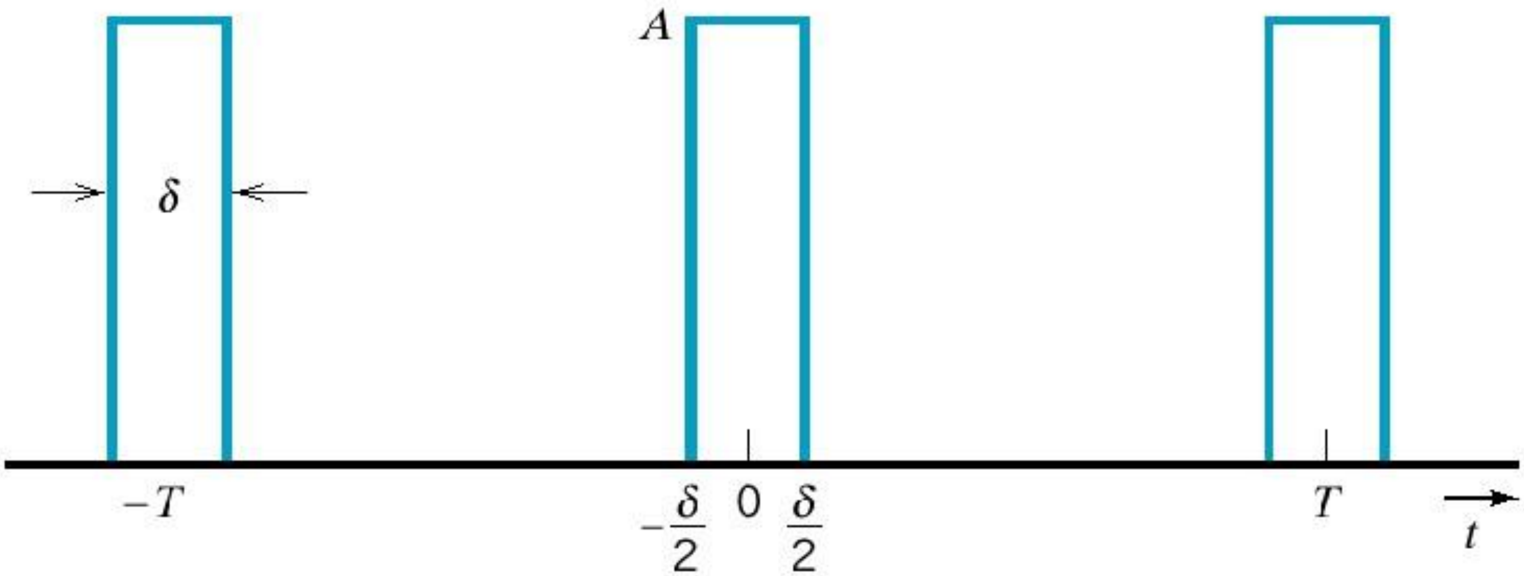


Phase spectrum

The Fourier Spectrum

The **Fourier Spectrum** is a graphical display of the amplitude and phase of the complex Fourier coe at the fundamental and harmonic frequencies.

Example



A periodic sequence of pulses each of width ·

The Fourier Spectrum

The Fourier coefficients are

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} A e^{-jn\omega_0 t}\,dt$$

For $n \neq 0$

$$C_n = \frac{A}{T}\int_{-\tau/2}^{\tau/2} e^{-jn\omega_0 t}\,dt$$

$$= \frac{-A}{jn\omega_0 T}\left[e^{-jn\omega_0 \tau/2} - e^{jn\omega_0 \tau/2}\right]$$

$$= \frac{2A}{n\omega_0 T}\sin\left(\frac{n\omega_0 \tau}{2}\right)$$

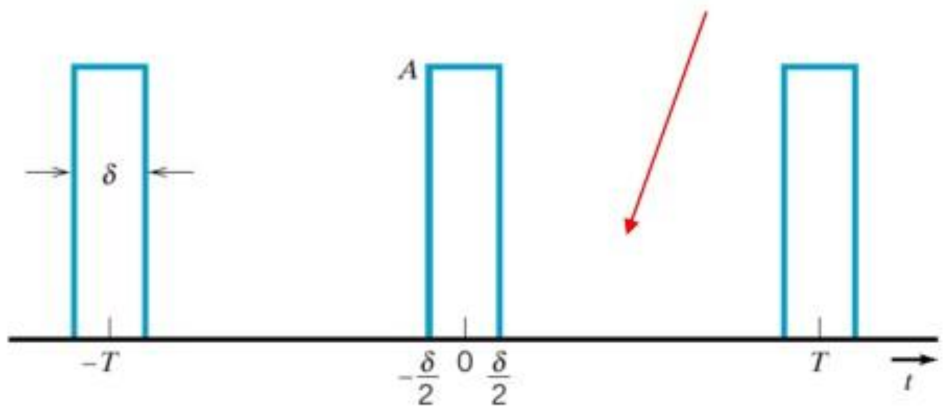
The Fourier Spectrum

$$C_n = \frac{A\delta}{T}\,\frac{\sin(n\omega_0\delta/2)}{(n\omega_0\delta/2)}$$

$$= \frac{A\delta}{T}\,\frac{\sin x}{x}$$

where $x = n\omega_0\delta/2$

For $n = 0$ $\quad C_0 = \frac{1}{T}\int_{-\delta/2}^{\delta/2} A\,dt = \frac{A\delta}{T}$

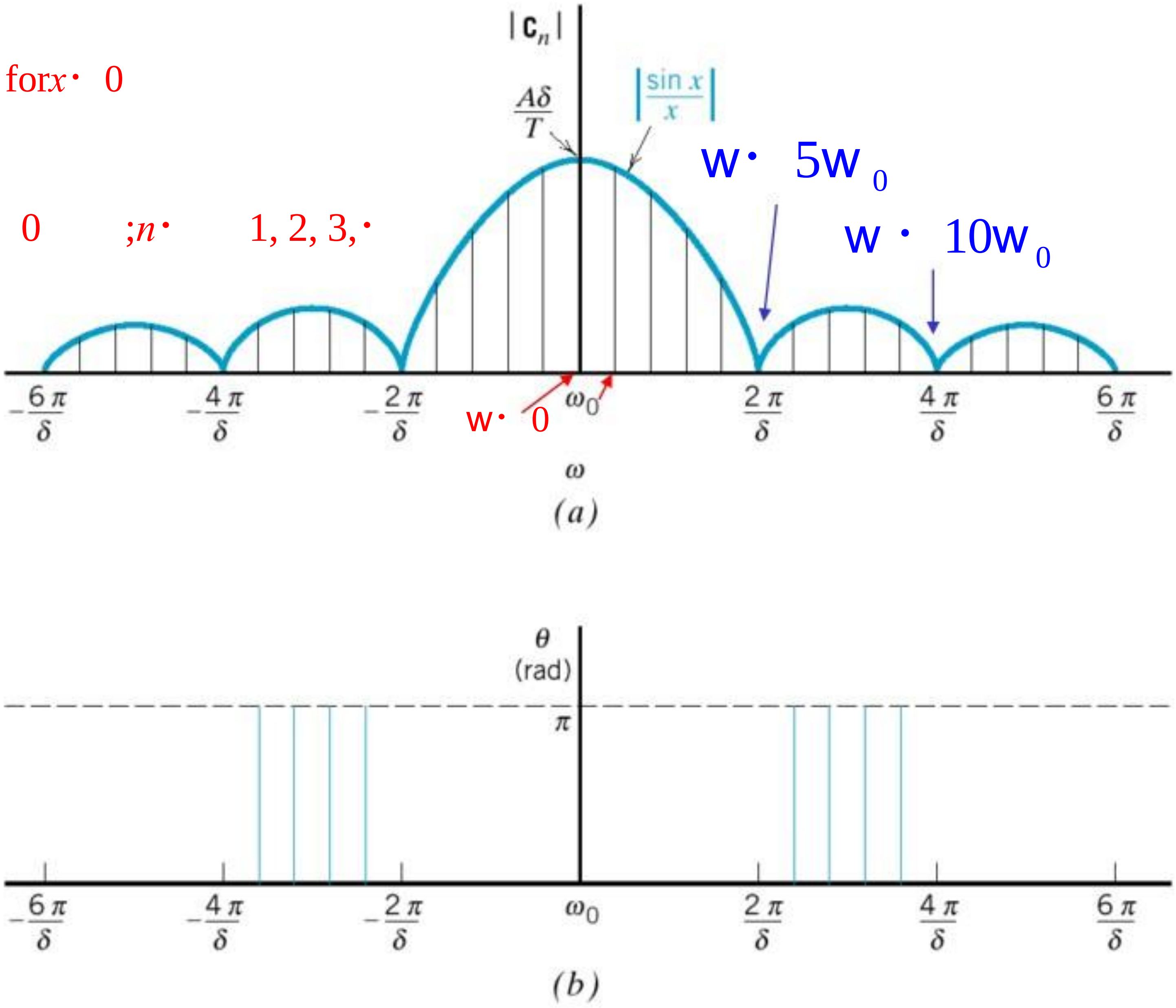


The Fourier Spectrum

L'Hopital's rule

$\frac{\sin x}{x}$ · 1 for x · 0

$\frac{\sin(n\cdot)}{n\cdot}$ · 0 ; n · 1, 2, 3, ·



The Truncated Fourier Series

A practical calculation of the Fourier series requires that we truncate the series to a ***finite*** number of terms.

$$f(t) \approx \sum_{n=-N}^{N} C_n e^{jn\omega_0 t} = S_N(t)$$

The error for *N* terms is

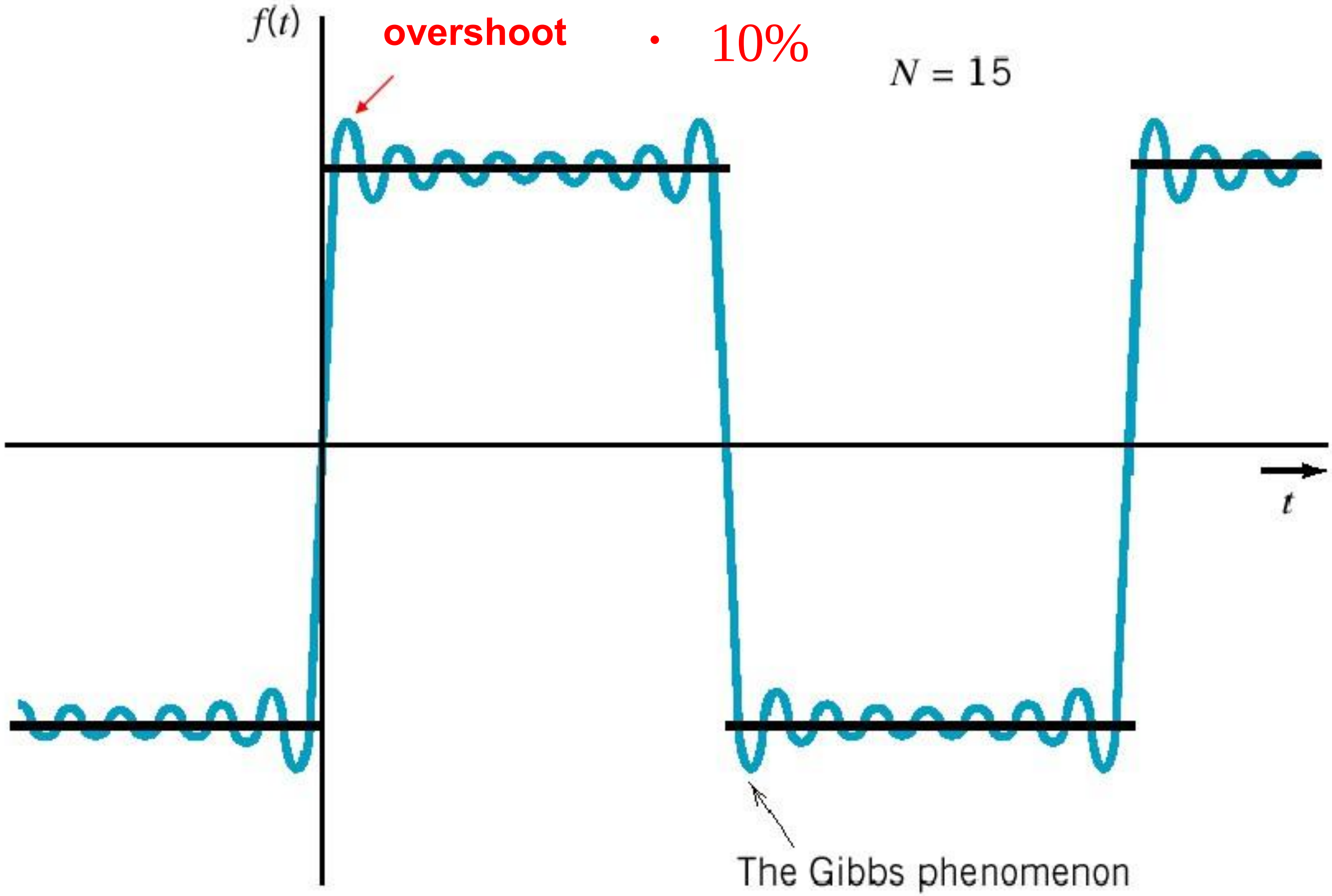
$$\varepsilon(t) = f(t) - S_N(t)$$

We use the mean-square error (MSE) defined as

$$MSE = \frac{1}{T}\int_0^T \varepsilon^2(t)\,dt$$

MSE is minimum when C_n = Fourier series' coefficients

The Truncated Fourier Series

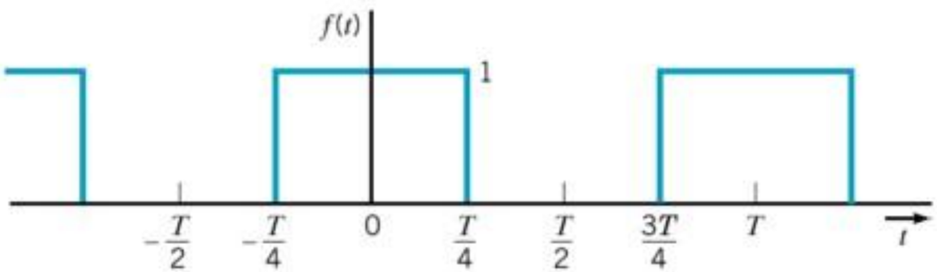


Circuits and Fourier Series

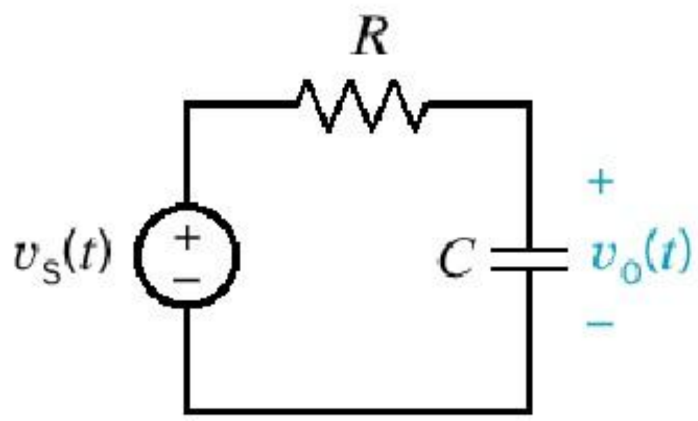
It is often desired to determine the response of a circuit excited by a periodic signal $v_S(t)$.

Example 15.8-1 An RC Circuit ***vo(t) = ?***

$$R \cdot\ 1 \cdot\ ,\ C \cdot\ 2\ \mathrm{F},\ T \cdot\ \cdot\ \ \mathrm{sec}$$



Example 15.3-1

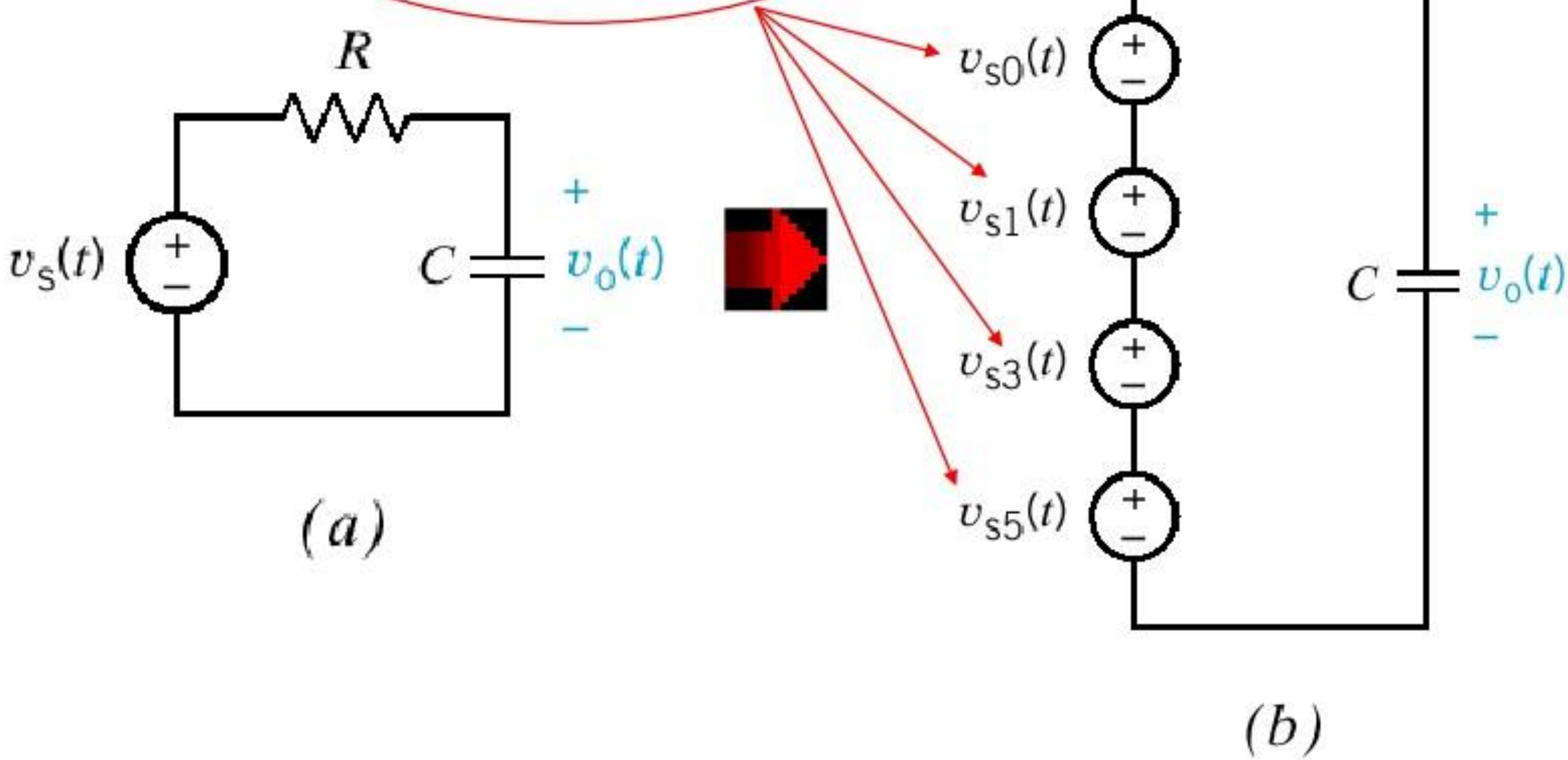


(a)

An *RC* circuit excited by a periodic voltage $v_S(t)$.

Circuits and Fourier Series

Each voltage source
is a term of the
Fourier series of $v_s(f)$.

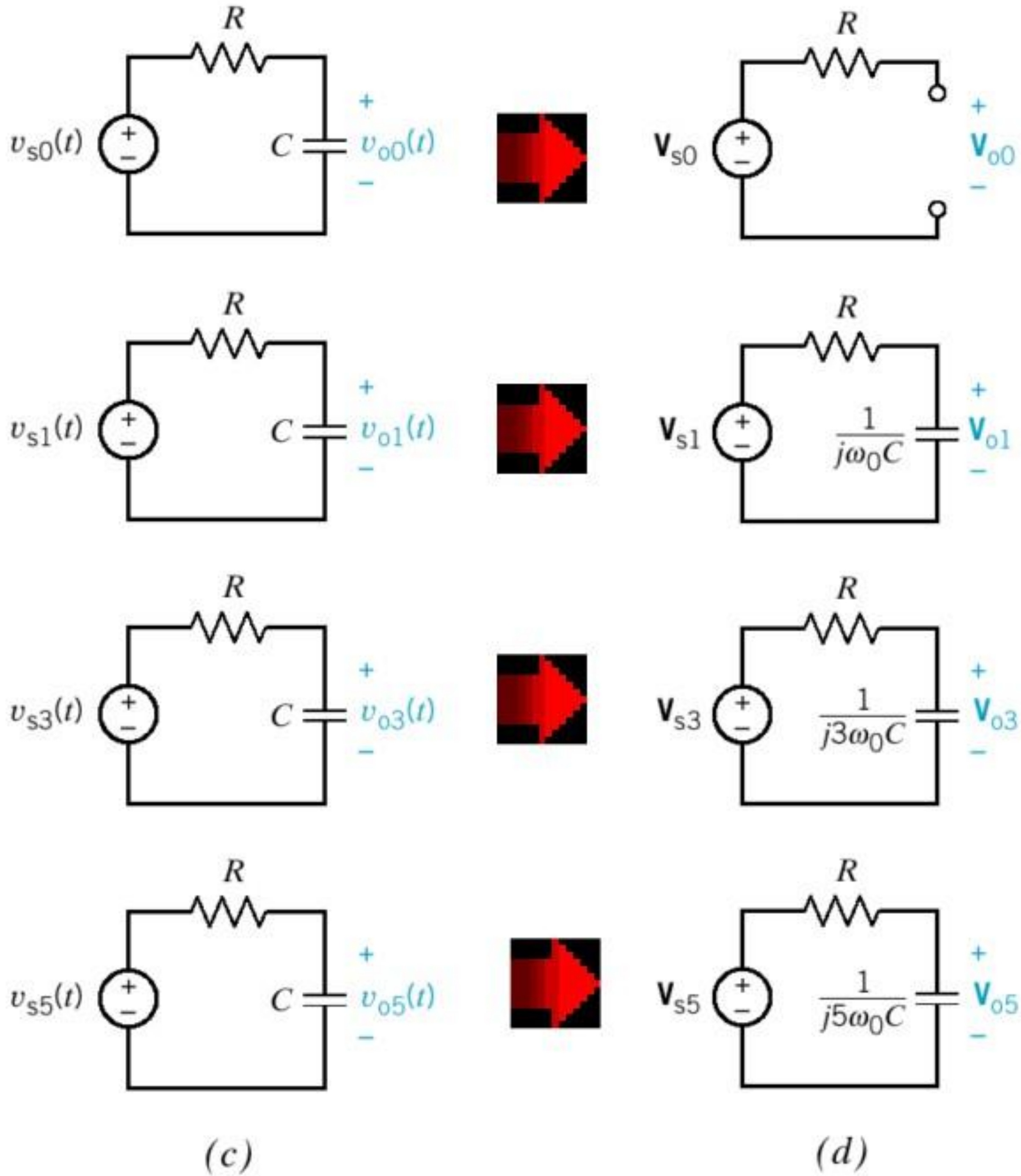


An equivalent circuit.

Example 5 (cont.)

Each input is a Sinusoid.

Using phasors to find steady-state responses to the sinusoids.



Example 5 (cont.)

$$v_s(t) = \frac{1}{2} + \sum_{n=1,odd}^{N} \frac{2(-1)^q}{n\pi} \cos n\omega_0 t$$

where $q = \frac{(n-1)}{2}$

The first 4 terms of $v_s(t)$ is

$$\omega_0 = 2 \text{ rad/s}$$

$$v_s(t) = \underbrace{\frac{1}{2}}_{v_{s0}(t)} + \underbrace{\frac{2}{\pi}\cos 2t}_{v_{s1}(t)} - \underbrace{\frac{2}{3\pi}\cos 6t}_{v_{s3}(t)} + \underbrace{\frac{2}{5\pi}\cos 10t}_{v_{s5}(t)}$$

The steady state response $v_o(t)$ can then be found using superposition.

$$v_o(t) = v_{o0}(t) + v_{o1}(t) + v_{o3}(t) + v_{o5}(t)$$

Example 5 (cont.)

The impedance of the capacitor is

$$\mathbf{Z}_C \cdot \frac{1}{jn\omega_0 C} \quad ; \text{for } n \cdot \; 0, 1, 3, 5, \cdot$$

We can find

$$\mathbf{V}_{on} \cdot \frac{\frac{1}{jn\omega_0 C}}{R \cdot \frac{1}{jn \quad C\omega_0}} \mathbf{V}_{sn} \quad ; \text{for } n \cdot \quad 0, 1, 3, 5, \cdot$$

$$\cdot \frac{}{1 \cdot \; jn\,\omega_0 CR}$$

4

Example 5 (cont.)

The steady-state response can be written as

$$v_{on}(t) \cdot \left|\mathbf{V}_{on}\right| \cos(n\omega_0 t \cdot \cdot \mathbf{V}_{on}$$

$$\cdot \frac{\left|\mathbf{V}_{sn}\right|}{\sqrt{1 \cdot 16n^2}} \cos(n\omega t_0 \cdot \cdot \mathbf{V}_{sn} \cdot \tan^{\cdot 1} 4n)$$

In this example we have

$$\left|\mathbf{V}_{s0}\right| \cdot \frac{1}{2}$$

$$\left|\mathbf{V}_{sn}\right| \cdot \frac{2}{n \cdot} \quad \text{for } n \cdot 1, 3, 5$$

$$\cdot \mathbf{V}_{sn} \cdot 0 \quad \text{for } n \cdot 0, 1, 3, 5$$

Example 5 (cont.)

$$v_{o0}(t) = \frac{1}{2}$$

$$v_{on}(t) = \frac{2}{n\pi\sqrt{1+16n^2}}\cos(n2t - \tan^{-1}4n) \quad ; \text{for } n = 1, 3, 5$$

$$v_{o1}(t) = 0.154\cos(2t - 76^\circ)$$

$$v_{o3}(t) = 0.018\cos(6t - 85^\circ)$$

$$v_{o5}(t) = 0.006\cos(10t - 87^\circ)$$

$$\therefore \quad v_o(t) = \frac{1}{2} + 0.154\cos(2t - 76^\circ) + 0.018\cos(6t - 85^\circ)$$

$$+ 0.006\cos(10t - 87^\circ)$$

Properties of Fourier Series

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

- Linearity

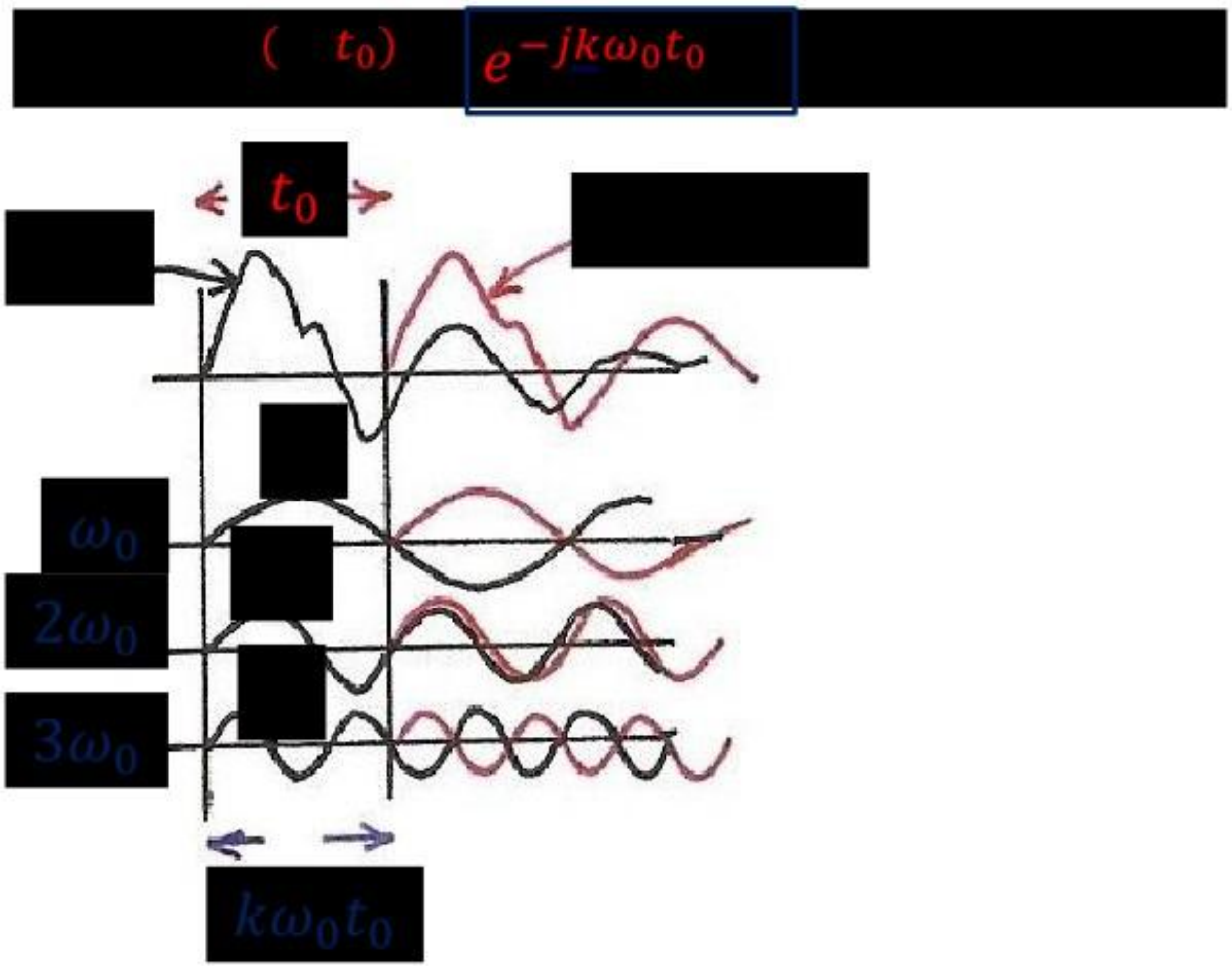
$$x(t) \overset{FS}{\longleftrightarrow} a_k,\quad y(t) \overset{FS}{\longleftrightarrow} b_k$$

$$Ax(t) + By(t) \overset{FS}{\longleftrightarrow} Aa_k + Bb_k$$

- Time Shift

$$x(t - t_0) \leftrightarrow e^{-jk\omega_0 t_0} a_k$$

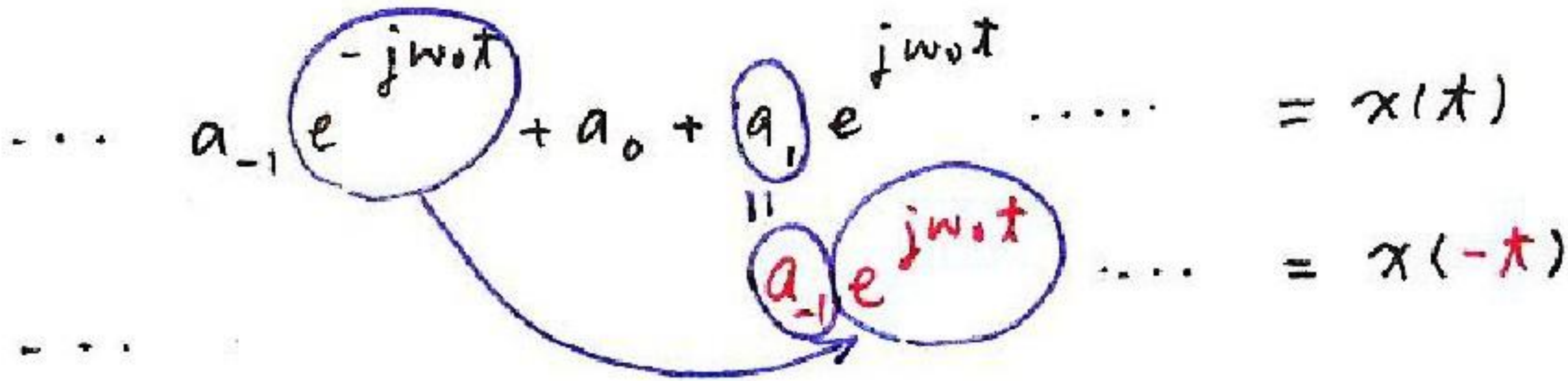
phase shift linear in frequency with amplitude unchanged



- Time Reversal

    $\cdot\ t\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ a_{\cdot\ k}$

    the effect of sign change for $x(t)$ and a_k are identical



unique representation for orthogonal basis

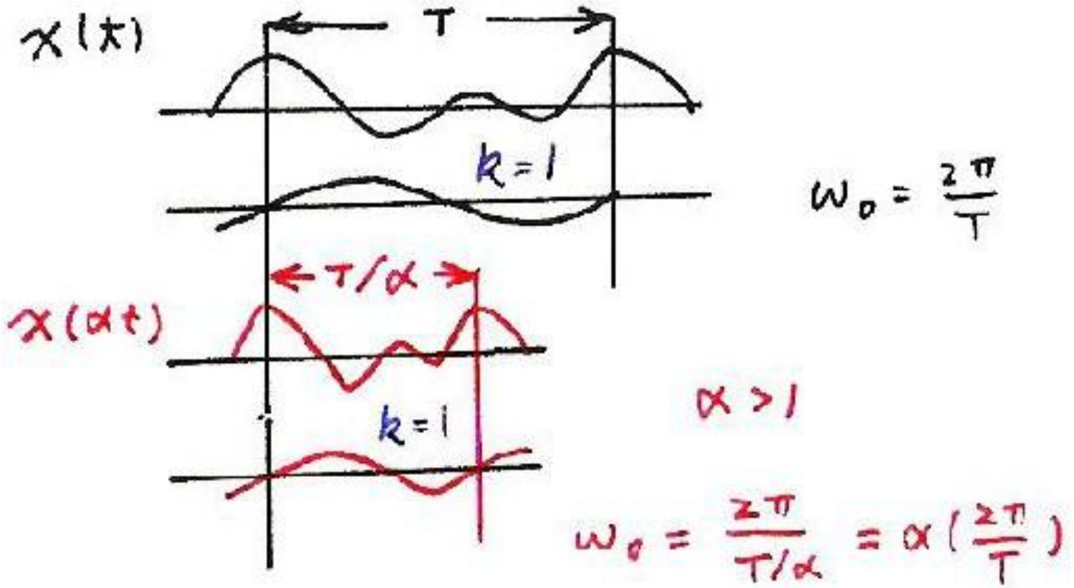
Time Scaling

- α :positive real number

$x(\alpha t)$:periodic with period T/α and fundamental frequency $\alpha\omega_0$

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

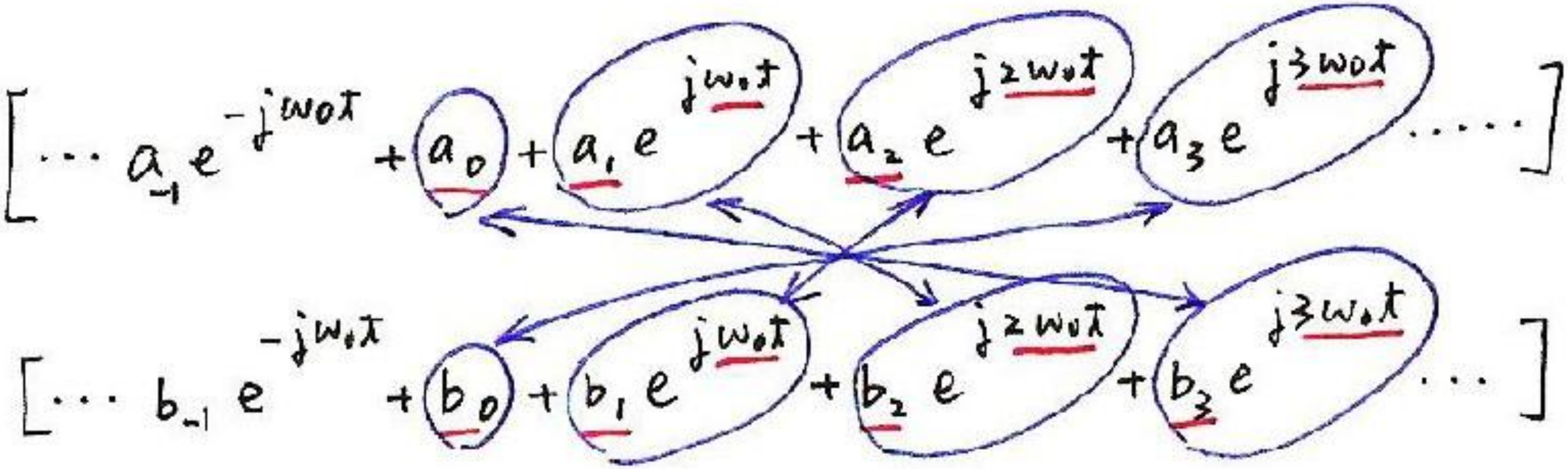
a_k unchanged, but $x(\alpha t)$ and each harmonic component are different



- Multiplication

$$x(t) \leftrightarrow a_k, \quad y(t) \leftrightarrow b_k$$

$$x(t)\,y(t) \leftrightarrow d_k = \sum_{j=-\infty}^{\infty} a_j b_{k-j} = a_k * b_k$$



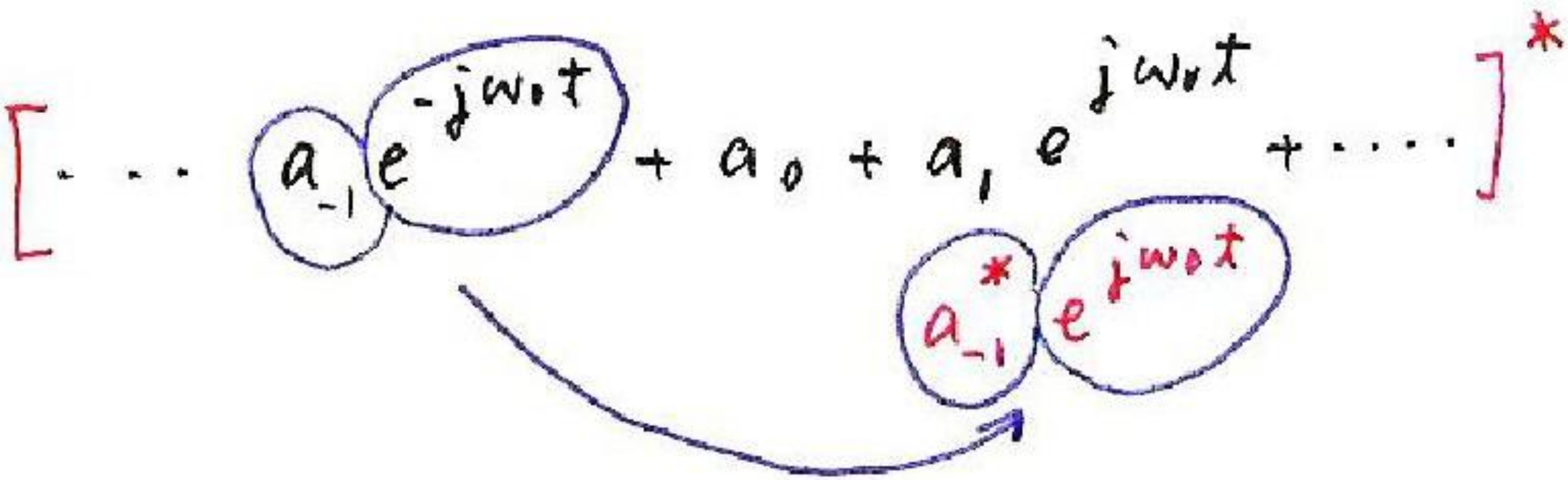
$$e^{j3\omega_0 t}$$

$$d_3 = \sum_j a_j b_{3-j}$$

- Conjugation

$$x^{*}(t) \overset{FS}{\longleftrightarrow} a^{*}_{-k}$$

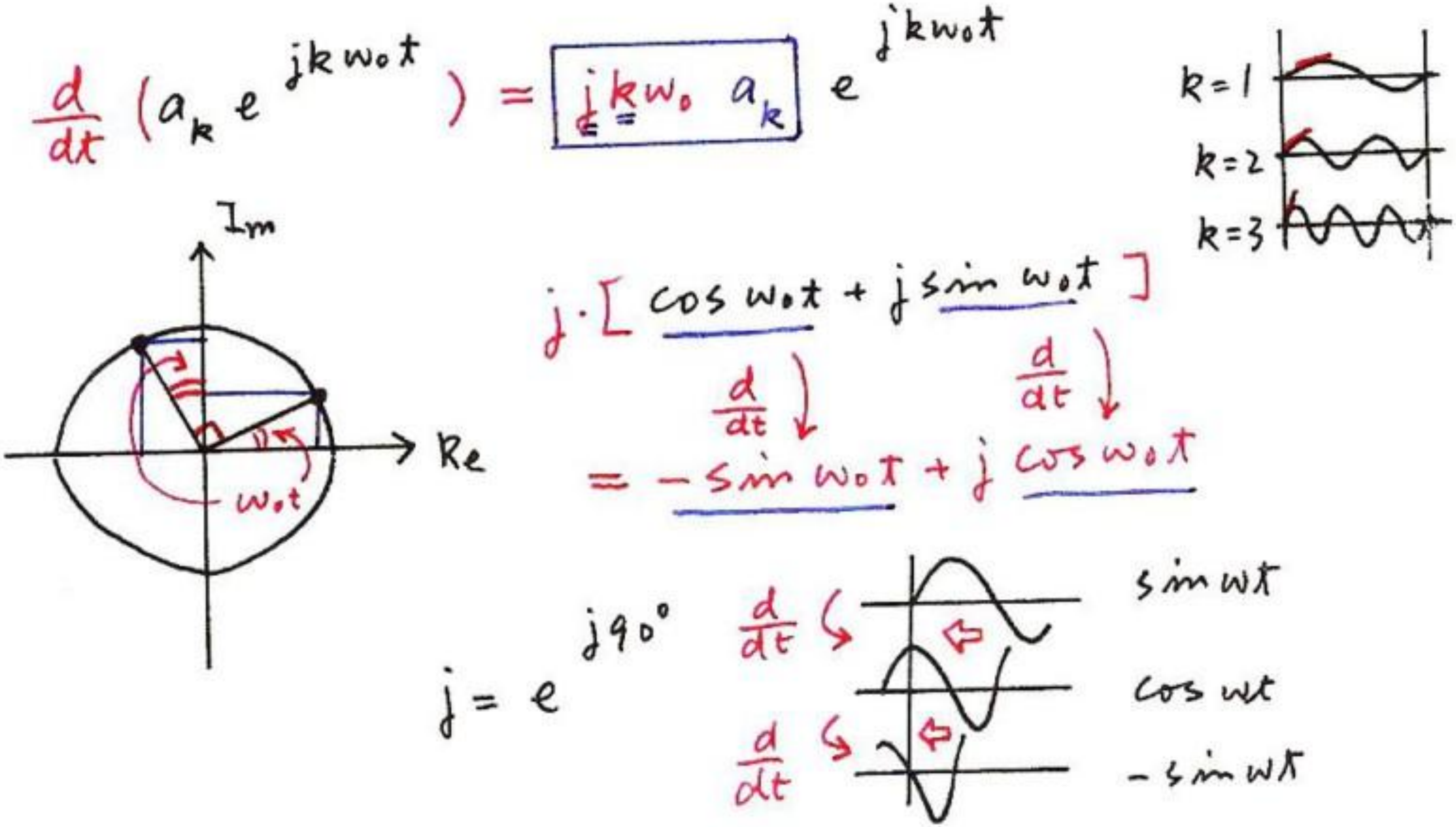
$$a_{-k} = a_k^{*}, \text{ if } x(t) \text{ real}$$



unique representation

- Differentiation

$$\frac{dxt\cdot}{dt} \cdot \cdot \cdot \cdot \; jk\text{w} \quad {}_0 a_k$$



- Parseval's Relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

total average power in a period T

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = |a_k|^2$$

average power in the k-th harmonic component in a period T

Continuous-Time Signal Analysis: The Fourier Transform

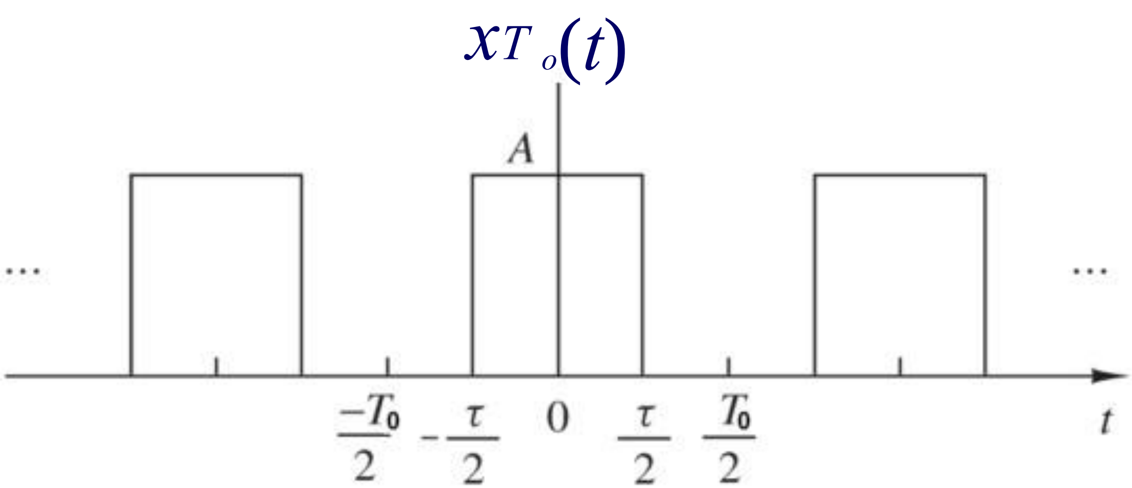
Chapter Outline

- Aperiodic Signal Representation by Fourier Integral
- Fourier Transform of Useful Functions
- Properties of Fourier Transform
- Signal Transmission Through LTIC Systems
- Ideal and Practical Filters
- Signal Energy
- Applications to Communications
- Data Truncation: Window Functions

Link between FT and FS

Fourier series (FS) allows us to represent periodic signal in term of sinusoidal or exponentials $e^{jn\omega_0 t}$.

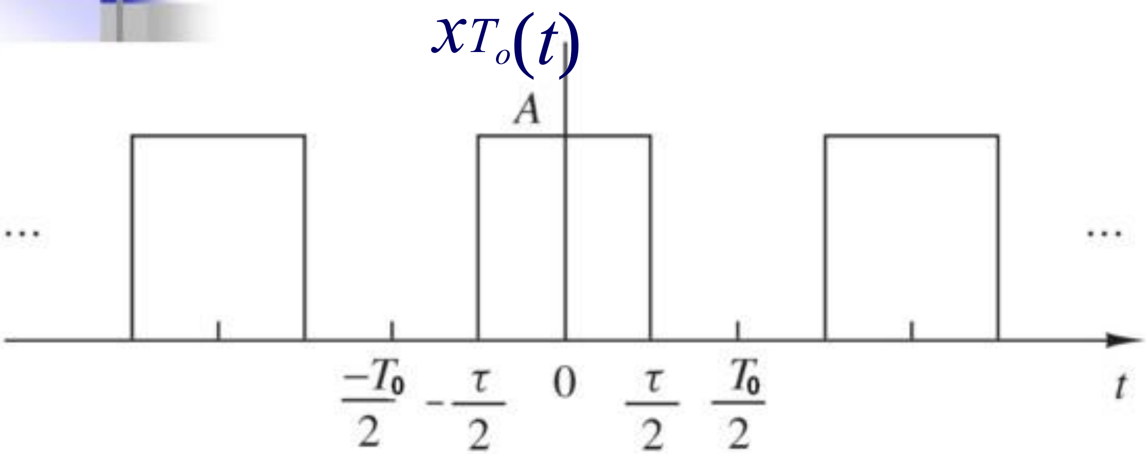
Fourier transform (FT) allows us to represent aperiodic (not periodic) signal in term of exponentials $e^{j\omega t}$.

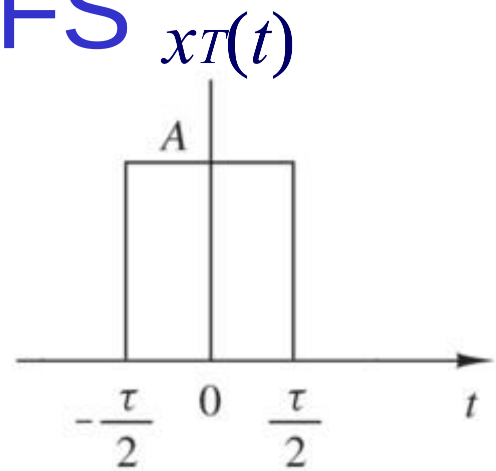


$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jn\omega_0 t} dt$$

Link between FT and FS

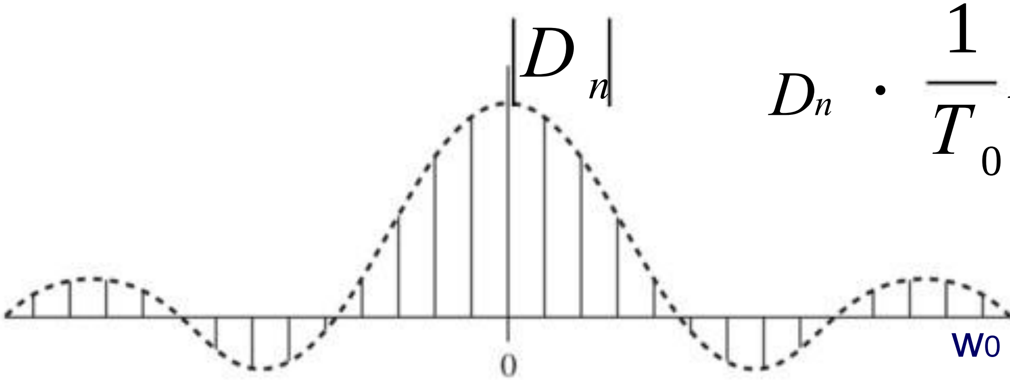




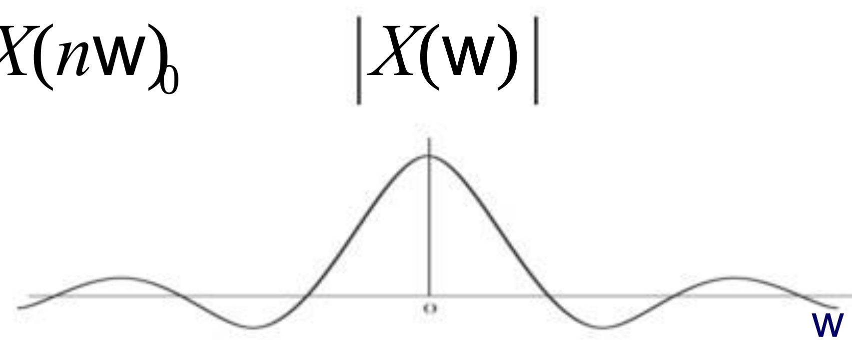
$$x(t) = \lim_{T_0 \to \infty} x_{T_0}(t)$$

$$T_0 \to \infty \Rightarrow w_0 \to 0$$

As T_0 gets larger and larger the fundamental frequency w_0 gets smaller and smaller so the spectrum becomes continuous.



$$D_n = \frac{1}{T_0} X(n w_0)$$



The Fourier Transform Spectrum

The Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

$$X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$$

The Amplitude (Magnitude) Spectrum

The Phase Spectrum

The amplitude spectrum is an even function and the phase is an odd function.

The Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

Example

Find the Fourier transform of $x(t) = e^{-at}u(t)$, the magnitude, and the spectrum

Solution:

$$X(\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t}\,dt = \frac{1}{a + j\omega} \quad \text{if } a > 0$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}} \qquad \angle X(\omega) = -\tan^{-1}(\omega / a)$$

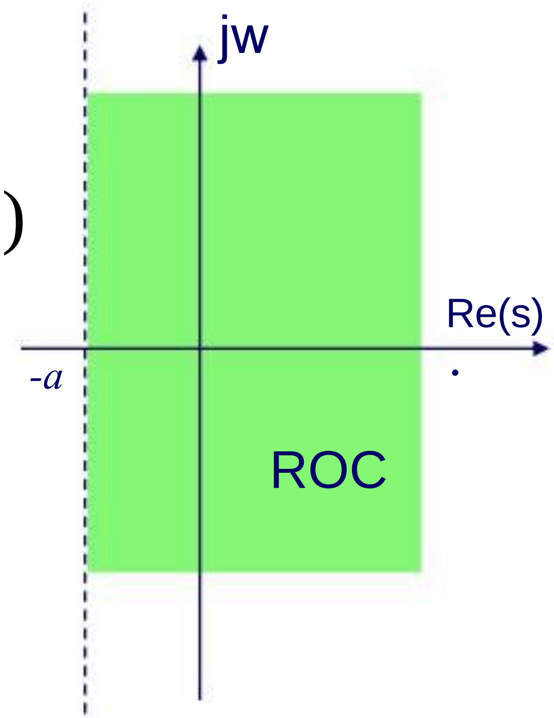
How does X(w) relates to X(s)?

$$X(s) = \int_0^{\infty} e^{-at} e^{-st}\,dt = -\frac{1}{a + s} e^{-(s+a)t}\Big|_0^{\infty}$$

$$X(s) = \frac{1}{a + s} \qquad \text{if } \mathrm{Re}(s) > -a$$

S-plane

$s = \sigma + j\omega$

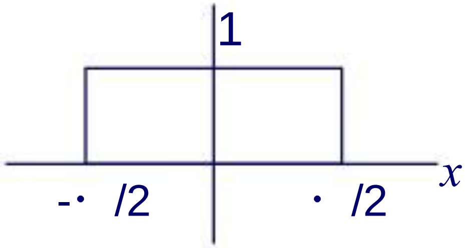


Since the jw-axis is in the region of convergence then FT exist.

Useful Functions

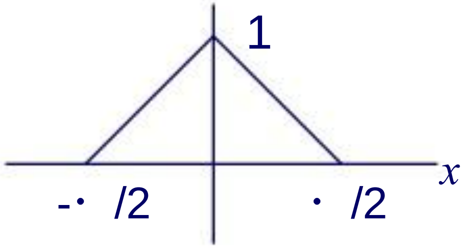
Unit Gate Function

$$rect\left(\frac{x}{\tau}\right) = \begin{cases} 0 & |x| > \tau/2 \\ 0.5 & |x| = \tau/2 \\ 1 & |x| < \tau/2 \end{cases}$$



Unit Triangle Function

$$\Delta\left(\frac{x}{\tau}\right) = \begin{cases} 0 & |x| \geq \tau/2 \\ 1 - 2|x|/\tau & |x| < \tau/2 \end{cases}$$



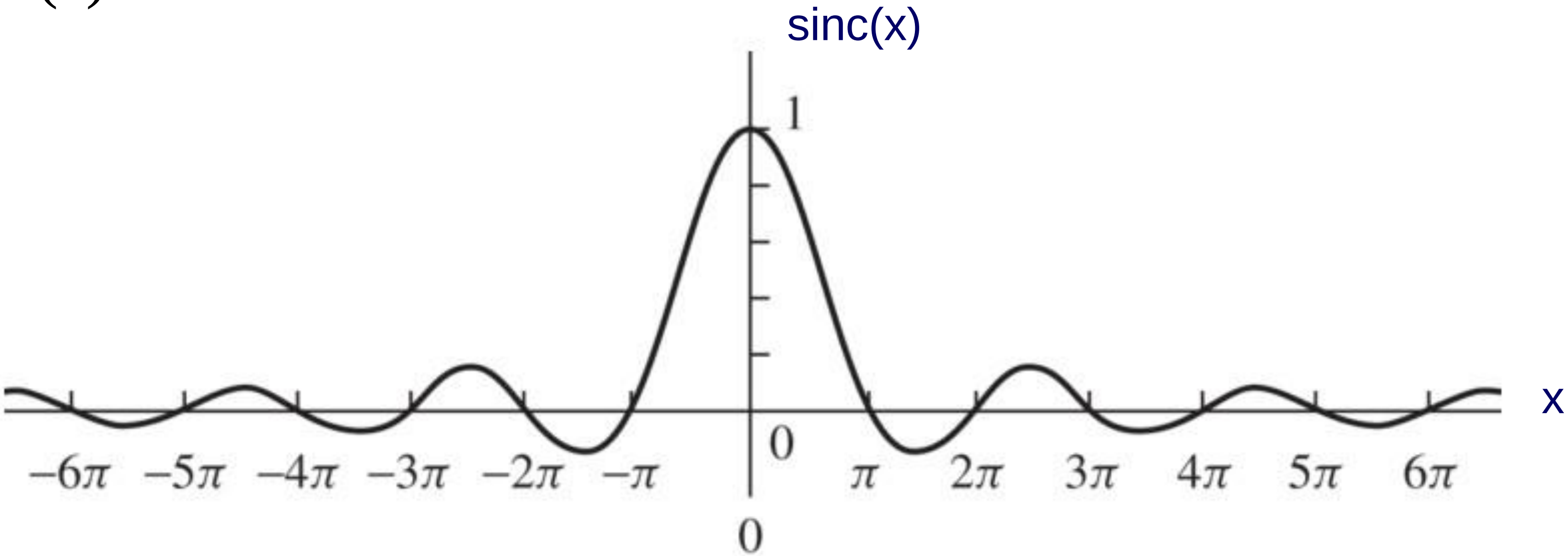
Useful Functions

Interpolation Function

$$\text{sinc}(x) \cdot \frac{\sin x}{x}$$

$$\text{sinc}(x) \cdot 0 \text{ for x} \cdot \cdot k \cdot$$

$$\text{sinc}(x) \cdot 1\text{for x} \cdot 0$$



Example

Find the FT, the magnitude, and the phase spectrum of x(t) = rect(t/τ).

Answer

$$X(w) = \int_{-\tau/2}^{\tau/2} rect(t/\tau)e^{-jwt}\,dt = \tau\,\mathrm{sinc}(w\tau/2)$$

What is the bandwidth of the above pulse?

The spectrum of a pulse extend from 0 to ∞. However, much of the spectrum is concentrated within the first lobe (w=0 to 2π/τ)

Examples

Find the FT of the unit impulse δ (t).

Answer

$$X(w) = \int_{-\infty}^{\infty} \delta(t) e^{-jwt} dt = 1$$

Find the inverse FT of δ (w).

Answer

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(w) e^{jwt} dw = \frac{1}{2\pi}$$

so thespectrumof a constantisan impulse

$$1 \Leftrightarrow 2\pi \delta(w)$$

Examples

Find the inverse FT of $\delta(\omega - \omega_0)$.

Answer

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

so the spectrum of a complex exponent is a shifted impulse

$$e^{j\omega t_0} \Leftrightarrow 2\pi\,\delta(\omega - \omega_0) \quad \text{and} \quad e^{-j\omega t_0} \Leftrightarrow 2\pi\,\delta(\omega + \omega_0)$$

Find the FT of the everlasting sinusoid $\cos(\omega_0 t)$.

Answer

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \Leftrightarrow \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

Examples

Find the FT of a periodic signal.

Answer

$$x(t) = \sum_{n=-\infty}^{n=\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = 2\pi / T_0$$

Take the FT of both side and use linearity property of FT

$$X(\omega) = 2\pi \sum_{n=-\infty}^{n=\infty} D_n \delta(\omega - n\omega_0)$$

Examples

Find the FT of the unit impulse train $\delta_{T_0}(t)$

Answer

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

$$X(\omega) = \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

TABLE Fourier Transforms

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	

TABLE Fourier Transforms

No.	$x(t)$	$X(\omega)$	
13	$\cos \omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$	
14	$\sin \omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$	
15	$e^{-at} \sin \omega_0 t\, u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$
16	$e^{-at} \cos \omega_0 t\, u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$
17	$\text{rect}\left(\frac{t}{\tau}\right)$	$\tau \,\text{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi} \text{sinc}\,(Wt)$	$\text{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2} \text{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi} \text{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$	$\omega_0 = \frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\, e^{-\sigma^2\omega^2/2}$	

Properties of the Fourier Transform

- *Linearity:*
- *Let* $x(t) \leftrightarrow X(\omega)$ *and* $y(t) \leftrightarrow Y(\omega)$

  then

$$x(t) + y(t) \leftrightarrow X(\omega) + Y(\omega)$$

- *Time Scaling:*
- *Let* $x(t) \leftrightarrow X(\omega)$

  then

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Compression in the time domain results in expansion in the frequency domain

Internet channel A can transmit 100k pulse/sec and channel B can transmit 200k pulse/sec. Which channel does require higher bandwidth?

Properties of the Fourier Transform

- *Time Reversal:*

- Let $x(t) \leftrightarrow X(w)$

then $x(-t) \leftrightarrow X(-w)$

Example: Find the FT of $e^{at}u(-t)$ and $e^{-a|t|}$

- *Left or Right Shift in Time:*

Time shift effects the phase and not the magnitude.

- Let $x(t) \leftrightarrow X(w)$

then $x(t - t_0) \leftrightarrow X(w)e^{-jwt_0}$

Example: if x(t) = sin(wt) then what is the FT of $x(t-t_0)$?

Example: Find the FT of $e^{-a|t-t_0|}$

Properties of the Fourier Transform

- *Multiplication by a Complex Exponential (Freq. Shift Property):*

- *Let*

then

$$x(t) \leftrightarrow X(\omega)$$

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

- *Multiplication by a Sinusoid (Amplitude Modulation):*

Let

$$x(t) \leftrightarrow X(\omega)$$

then

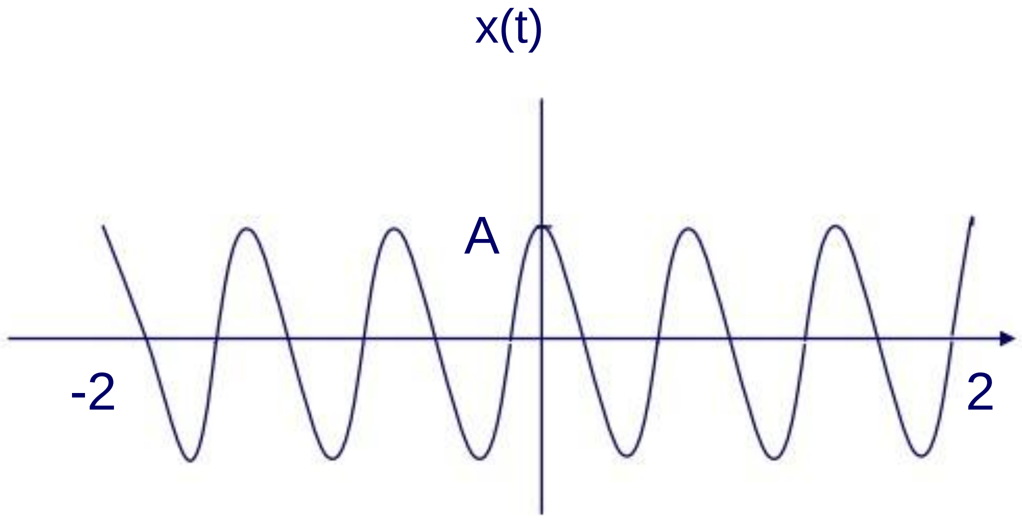
$$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]$$

$\cos\omega_0 t$ is the carrier, x(t) is the modulating signal (message),
x(t) $\cos\omega_0 t$ is the modulated signal.

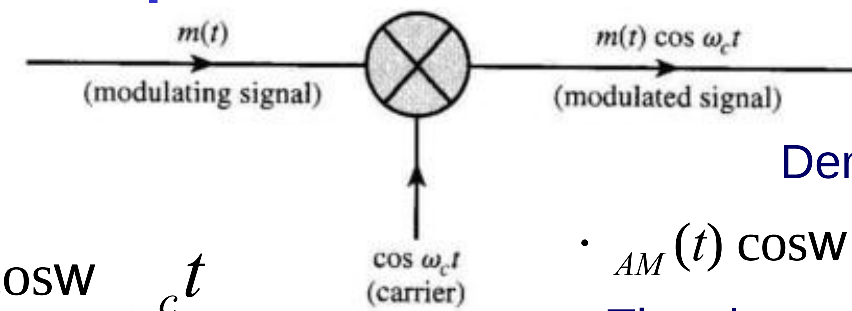
Example: Amplitude Modulation

Example: Find the FT for the signal

$$x(t) \cdot \quad rect(t/4)\cos 10t$$



Amplitude Modulation



(a)

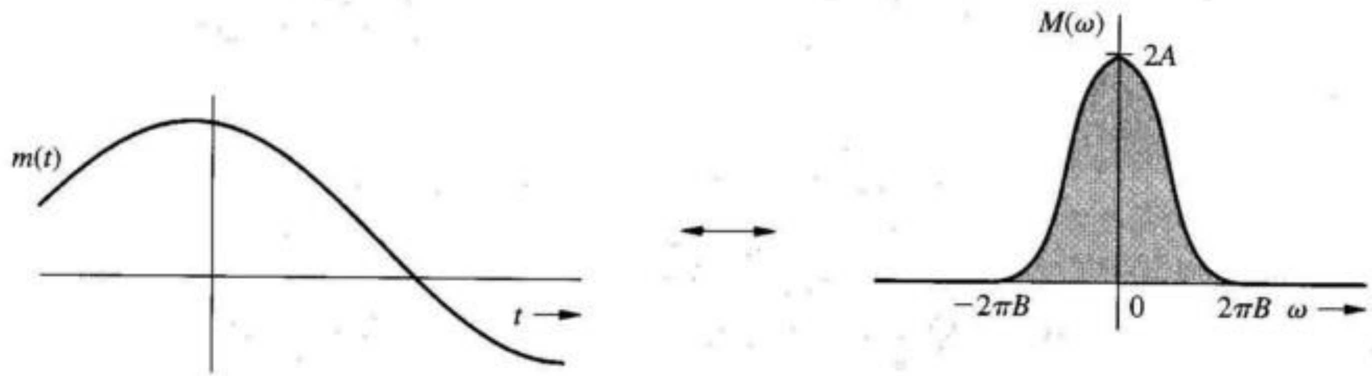
Modulation

- $_{AM}(t) \cdot \quad m(t)\cos \mathrm{w}_c t$

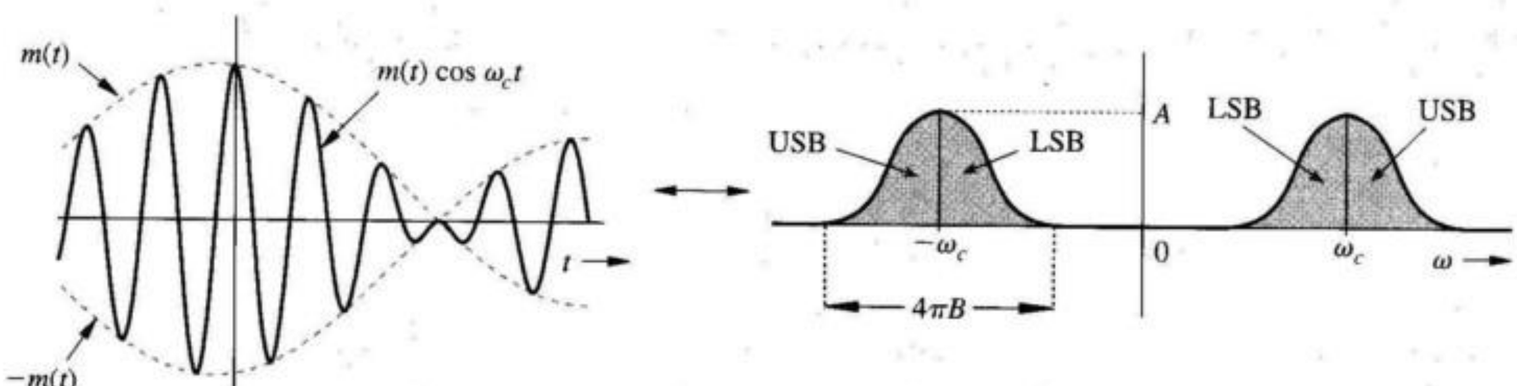
Demodulation

- $_{AM}(t)\cos \mathrm{w}_c t \cdot \; 0.5m(t)[1 \cdot \; \cos 2\mathrm{w}_c t]$

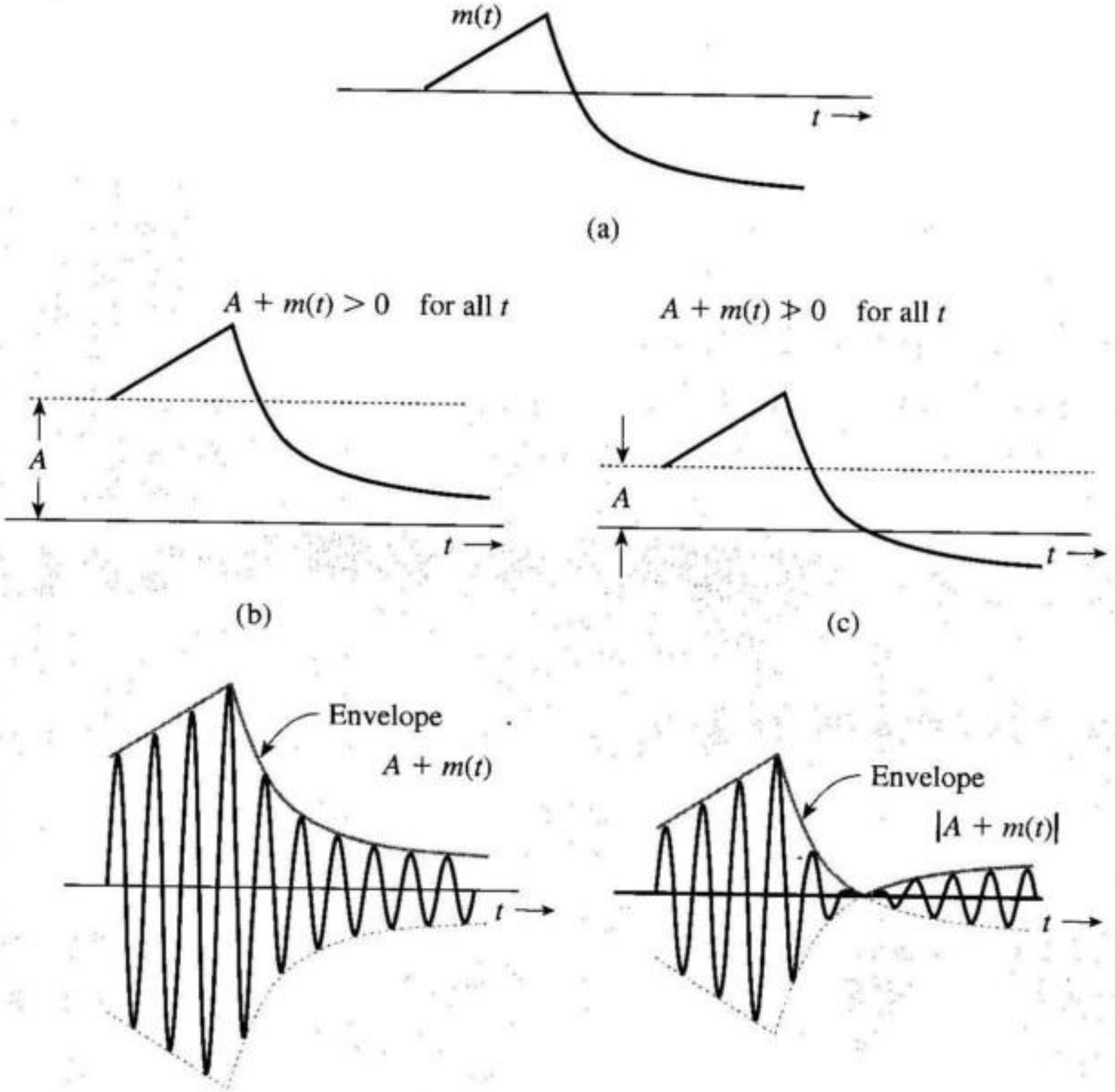
Then lowpass filtering



(b)



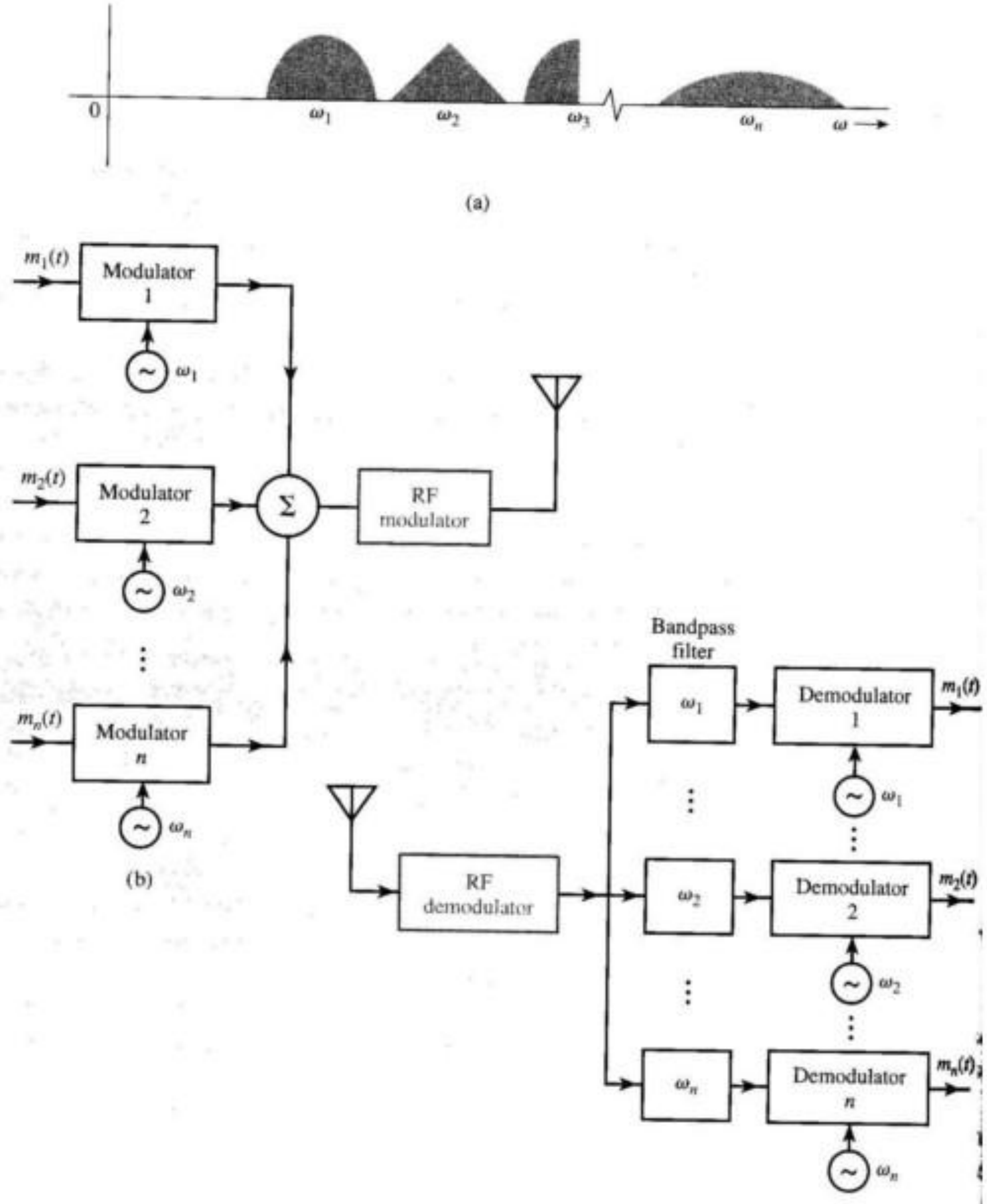
Amplitude Modulation: Envelope Detector



Applic. of Modulation: Frequency-Division Multiplexing

1- Transmission of different signals over different bands

2- Require smaller antenna



Properties of the Fourier Transform

- *Differentiation in the Frequency Domain:*
- *Let*

$$x\bullet \quad t\bullet \quad \cdot \quad X\bullet \quad w\bullet$$

then

$$t^n x\ t)\cdot \quad (j) \qquad {}^n \frac{d^n}{dw^n} X(w)$$

- *Differentiation in the Time Domain:*

Let

$$x\bullet \quad t\bullet \quad \cdot \quad wX\bullet \qquad \cdot$$

then

$$\frac{d^n}{dt^n} x\ t)\cdot \quad (\qquad jw)^n X(w)$$

Example: Use the time-differentiation property to find the Fourier Transform of the triangle pulse x(t) = · (t/·)

Properties of the Fourier Transform

- *Integration in the Time Domain:*

Let

$$x\bullet \quad t\bullet \quad \cdot \; X\bullet \quad \mathrm{w}\bullet$$

Then

$$\bullet^{t}_{\cdot\ \cdot}\; x(\cdot\ \ )d\bullet \quad \cdot \qquad \frac{1}{j\mathrm{w}}X(\mathrm{w})\cdot \quad \cdot \quad X(0)\cdot \quad (\mathrm{w})$$

- *Convolution and Multiplication in the Time Domain:*

Let

$$x\ t\bullet \quad \cdot \; X\bullet \quad \mathrm{w}\bullet$$

$$y\ t\bullet \quad \cdot \quad Y\bullet \quad \mathrm{w}\bullet$$

Then

$$x(t)\ \cdot \quad y(t)\ \cdot \quad X(\mathrm{w})Y(\mathrm{w})$$

$$x_1 t) x(t) \cdot_2 \qquad \frac{1}{2\cdot} X_1(\mathrm{w})\cdot \quad X_2(\mathrm{w})$$

Frequency convolution

Example

Find the system response to the input $x(t) = e^{-at} u(t)$ if the system impulse response is $h(t) = e^{-bt} u(t)$.

Properties of the Fourier Transform

- *Parseval's Theorem*: since $x(t)$ is non-periodic and has FT $X(\omega)$, then it is an energy signals:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Real signal has even spectrum $X(\omega) = X(-\omega)$,

$$E = \frac{1}{\pi}\int_{0}^{\infty} |X(\omega)|^2 d\omega$$

Example

Find the energy of signal $x(t) = e^{-at} u(t)$. Determine the frequency w so that the energy contributed by the spectrum components of all frequencies below w is 95% of the signal energy E_X.

Answer: w=12.7a rad/sec

Properties of the Fourier Transform

- *Duality (Similarity):*
- *Let*

$$x(t) \leftrightarrow X(\omega)$$

then

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

TABLE Fourier Transform Operations

Operation	$x(t)$	$X(\omega)$
Scalar multiplication	$kx(t)$	$kX(\omega)$
Addition	$x_1(t) + x_2(t)$	$X_1(\omega) + X_2(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Scaling (a real)	$x(at)$	$\frac{1}{\|a\|} X\left(\frac{\omega}{a}\right)$
Time shifting	$x(t - t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting (ω_0 real)	$x(t)e^{j\omega_0 t}$	$X(\omega - \omega_0)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Time differentiation	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
Time integration	$\int_{-\infty}^{t} x(u)\,du$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

Sampling Theorem

A real signal whose spectrum is bandlimited to B Hz [$X(\omega)=0$ for $|\omega| > 2\pi B$] can be reconstructed exactly from its samples taken uniformly at a rate $f_s > 2B$ samples per second. When $f_s = 2B$ then f_s is the Nyquist rate.

$$\bar{x}(t) = x(nT) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT)$$

$$\bar{x}(t) = x(nT) = x(t)\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$\bar{X}(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

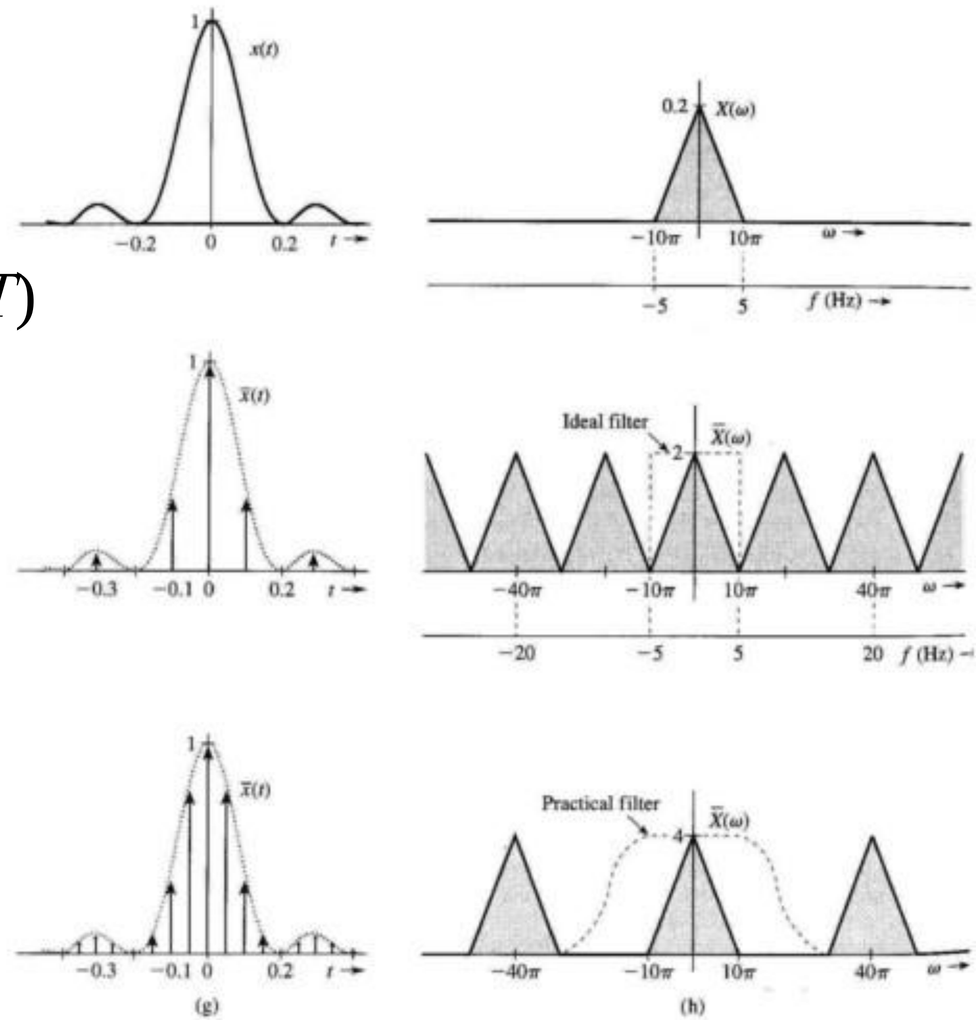


Figure 8.2 Effects of undersampling and oversampling.

Reconstructing the Signal from the Samples

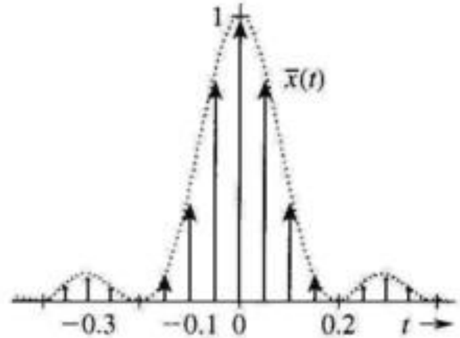
$$X(\omega) = H(\omega)\,\overline{X}(\omega)$$

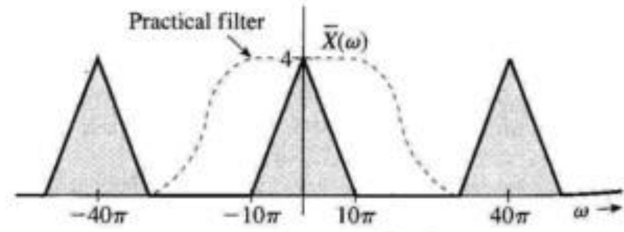
$$x(t) = h(t) * x(nT)$$

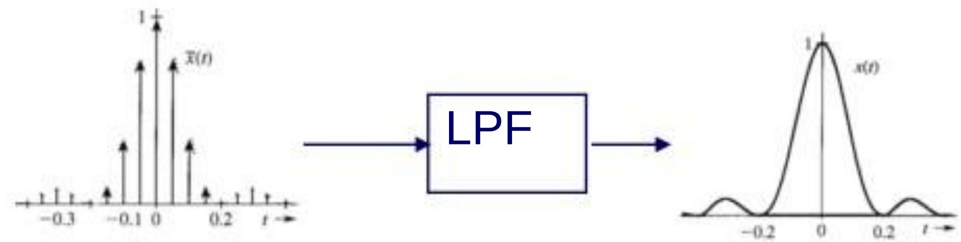
$$x(t) = h(t) * \sum_{n} x(nT)\,\delta(t - nT)$$

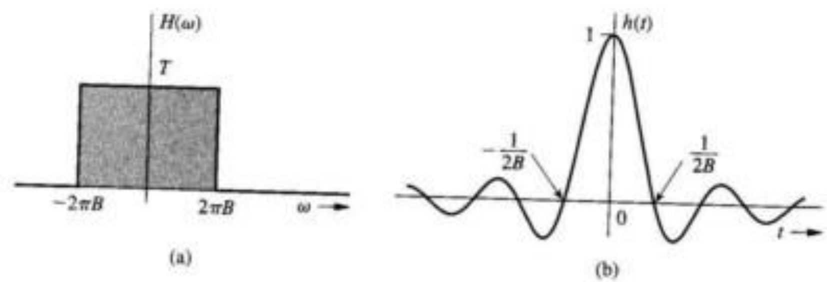
$$x(t) = \sum_{n} x(nT)\,h(t - nT)$$

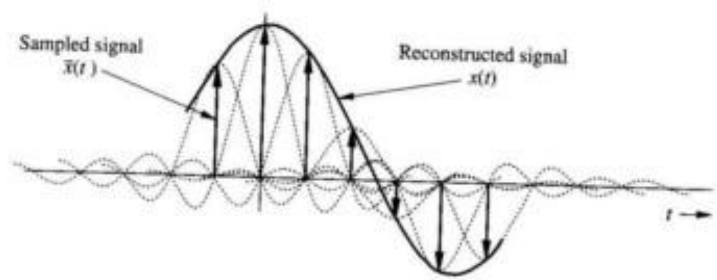
$$x(t) = \sum_{n} x(nT)\,\mathrm{sinc}(2\pi B(t - nT))$$











Example

Determine the Nyquist sampling rate for the signal

x(t) = 3 + 2 cos(10•) + sin(30•).

Solution

The highest frequency is f_{max} = 30• /2•   = 15 Hz

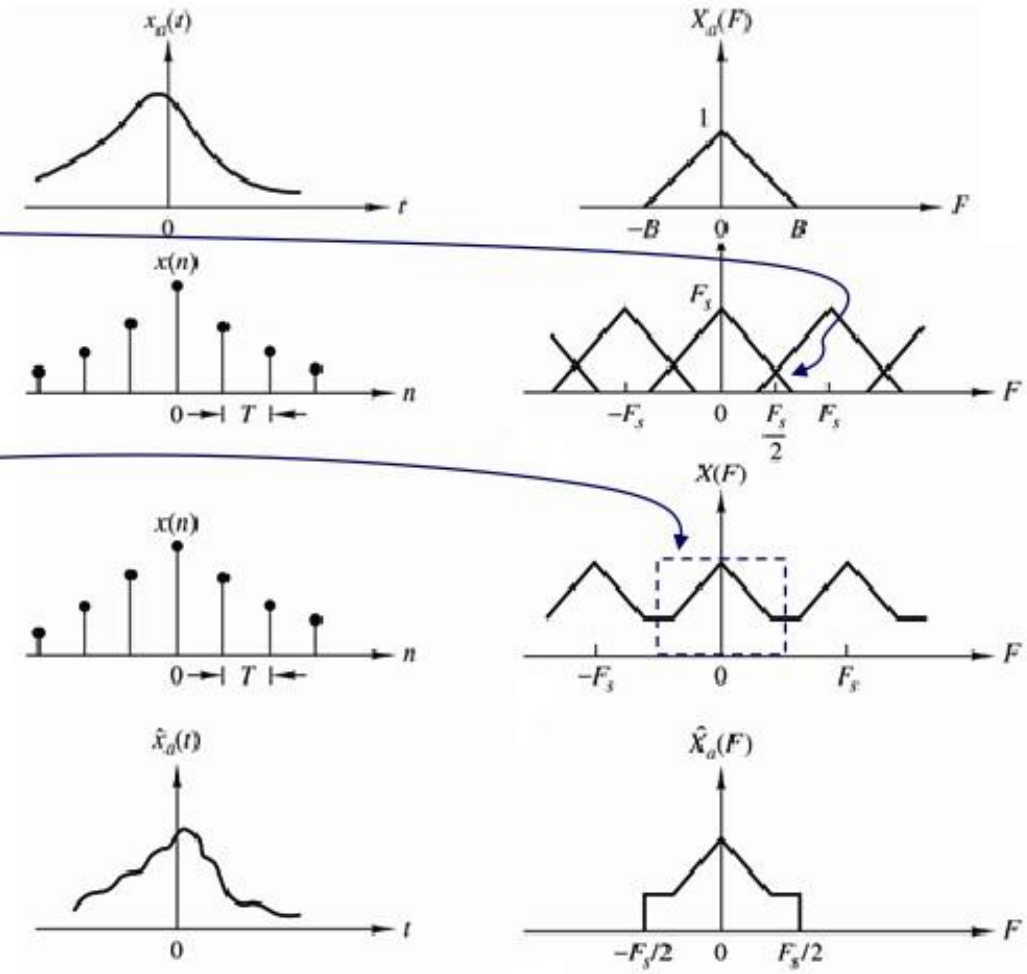
The Nyquist rate = 2 f_{max} = 2*15 = 30 sample/sec

Aliasing

If a continuous time signal is sampled below the Nyquist rate then some of the high frequencies will appear as low frequencies and the original signal can not be recovered from the samples.

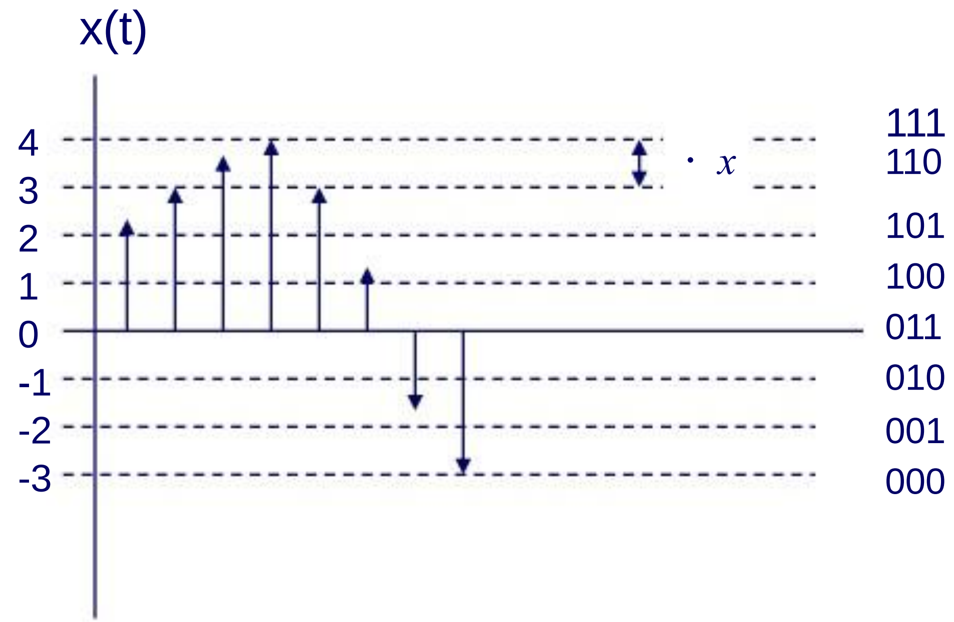
Frequency above $F_s/2$ will appear (aliased) as frequency below $F_s/2$

LPF
With cutoff frequency
Fs/2



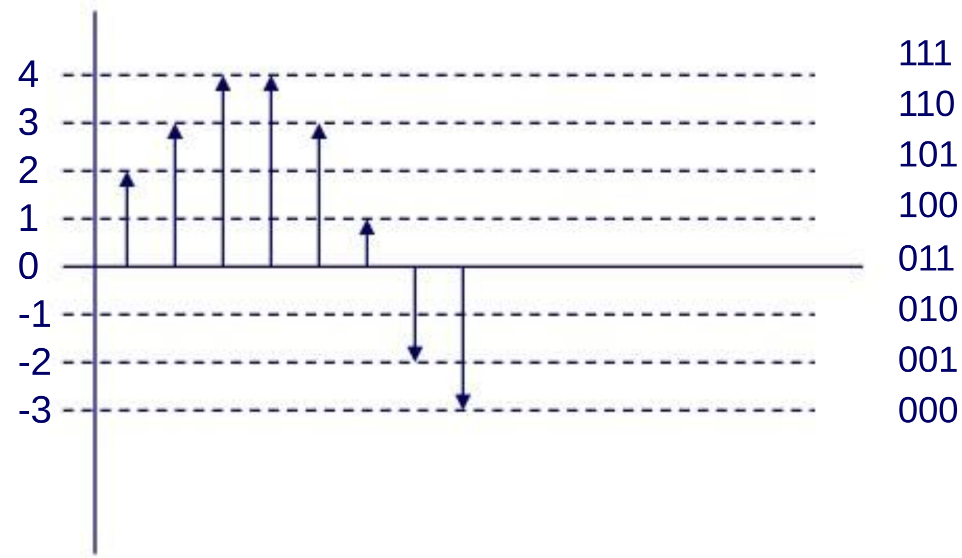
Quantization & Binary Representation

$$L \cdot 2^{n}$$



L : number of levels
n : Number of bits
Quantization error = · x/2

$$\cdot \; x \cdot \; \frac{x_{\max} \cdot x_{\min}}{L \cdot 1}$$



Example

A 5 minutes segment of music sampled at 44000 samples per second. The amplitudes of the samples are quantized to 1024 levels. Determine the size of the segment in bits.

Solution

# of bits per sample = ln(1024) { remember $L=2^n$ }

n = 10 bits per sample

# of bits = 5 * 60 * 44000 * 10 = 13200000 = 13.2 Mbit

Discrete-Time Processing of Continuous-Time Signals

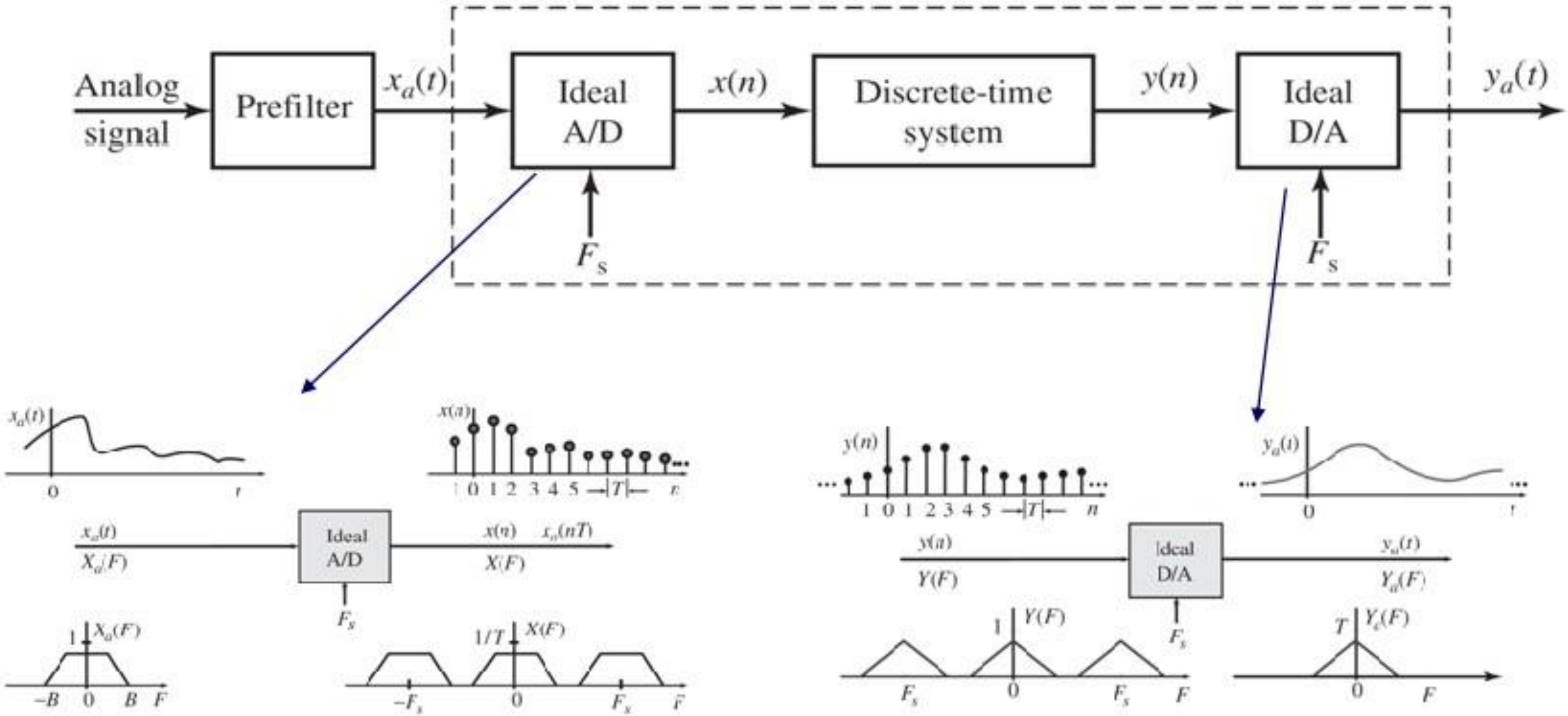
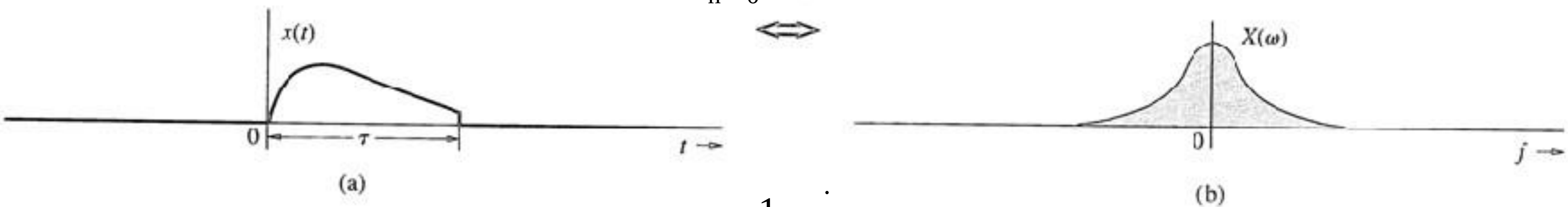


Figure Characteristics of an ideal A/D converter in the time and frequency domains.

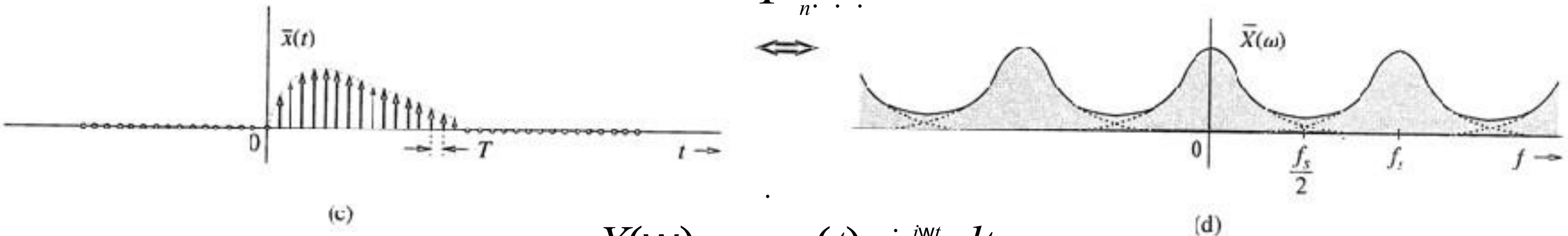
Figure Characteristics of an ideal D/A converter in the time and frequency domains.

Discrete Fourier Transform

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$



$$X(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

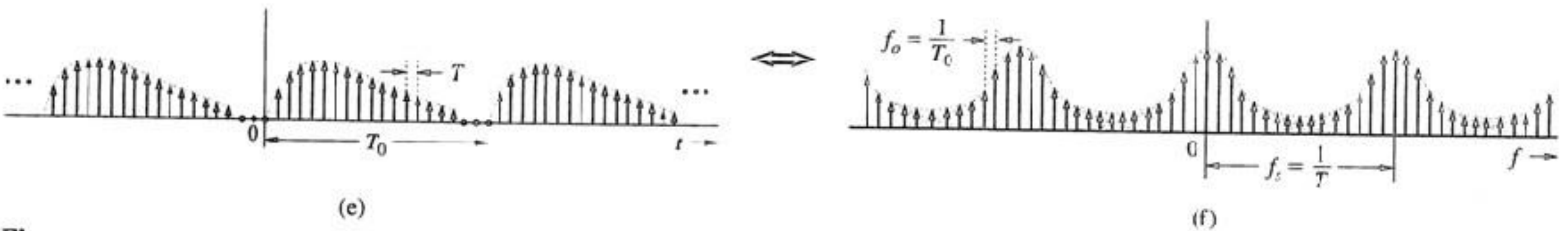
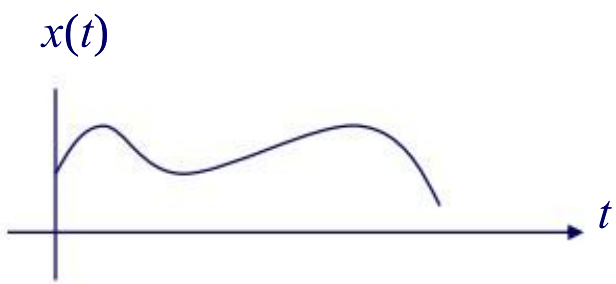


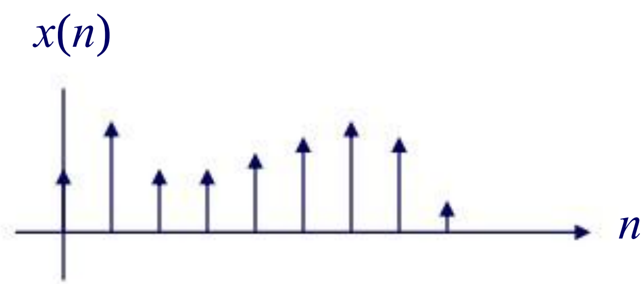
Figure Relationship between samples of $x(t)$ and $X(\omega)$.

Link between Continuous and Discrete

$x(t)$ —— Sampling Theorem ——→ $x(n)$



Continuous



Discrete

$x(t)$ —— Laplace Transform ——→ $X(s)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$x(t)$ —— Fourier Transform ——→ $X(j\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$x(n)$ —— z Transform ——→ $X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$x(n)$ —— Discrete Fourier Transform ——→ $X(k)$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi k}{N}n}$$

HILBERT TRANSFORM

- Fourier, Laplace, and z-transforms **change from the time-domain** representation of a signal **to the frequency-domain** representation of the signal
- The resulting two signals are **equivalent representations of the same signal** in terms of time or frequency
- In contrast, The Hilbert transform does **not involve a change of domain**, unlike many other transforms

HILBERT TRANSFORM

- Strictly speaking, **the Hilbert transform is not a transform** in this sense

  - First, the result of a Hilbert transform is **not equivalent to the original signal**, rather it is a completely different signal

  - Second, the Hilbert transform does **not involve a domain change**, i.e., the Hilbert transform of a signal x(t) is another signal denoted by $x(t)$ in the same domain (i.e.,time domain)

HILBERT TRANSFORM

- The Hilbert transform of a signal x(t) is a signal $x(t)$ whose frequency components lag the frequency components of x(t) by 90·
  - $x(t)$ has exactly the same frequency components present in x(t) with the same amplitude-except there is a 90· phase delay
  - The Hilbert transform of x(t) = Acos(2· f_0t + · **)** is Acos(2· f_0t + · **-** 90·) = Asin(2· f_0t + · **)**

HILBERT TRANSFORM

- A delay of · /2 at all frequencies
  - $e^{j2\cdot f_0 t}$ will become $e^{j2\cdot f_0 t \cdot \frac{\cdot}{2}} \cdot \cdot je^{j2\cdot f_0 t}$
  - $e^{-j2\cdot f_0 t}$ will become $e^{\cdot j(2\cdot f_0 t \cdot \frac{\cdot}{2})} \cdot je^{j2\cdot f_0 t}$
    - At positive frequencies, the spectrum of the signal is multiplied by -*j*
    - At negative frequencies, it is multiplied by +*j*
  - This is equivalent to saying that the spectrum (Fourier transform) of the signal is multiplied by

    -*j*sgn(*f*).

HILBERT TRANSFORM

- Assume that x(t) is real and has no DC component : $X(f)|_{f=0} = 0$,

then $$F[x(t)] = -j\,\mathrm{sgn}(f)X(f)$$

$$F^{-1}[-j\,\mathrm{sgn}(f)] = \frac{1}{\pi t}$$

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\,d\tau$$

- The operation of the Hilbert transform is equivalent to a convolution, i.e., filtering

Example

- Determine the Hilbert transform of the signal x*(t)* = 2sinc(2t)
- ***Solution***
  - We use the frequency-domain approach . Using the scaling property of the Fourier transform, we have

$$\mathcal{F}[x(t)] = 2 \times \frac{1}{2}\Pi\left(\frac{f}{2}\right) = \Pi\left(f + \frac{1}{2}\right) + \Pi\left(f - \frac{1}{2}\right)$$

  - In this expression, the first term contains all the negative frequencies and the second term contains all the positive frequencies
  - To obtain the frequency-domain representation of the Hilbert transform of ***x(t),*** we use the relation $\mathcal{F}[\hat{x}(t)] = -j\,\mathrm{sgn}(f)\mathcal{F}[x(t)]$, which results in

$$\mathcal{F}[\hat{x}(t)] = j\Pi\left(f + \frac{1}{2}\right) - j\Pi\left(f - \frac{1}{2}\right)$$

  - Taking the inverse Fourier transform, we have

$$\hat{x}(t) = je^{-j\pi t}\mathrm{sinc}(t) - je^{j\pi t}\mathrm{sinc}(t) = -j(e^{j\pi t} - e^{-j\pi t})\mathrm{sinc}(t)$$

$$= -j \times 2j\sin(\pi t)\mathrm{sinc}(t) = 2\sin(\pi t)\mathrm{sinc}(t)$$

HILBERT TRANSFORM

- Obviously performing the Hilbert transform on a signal is equivalent to a 90· phase shift in all its frequency components
- Therefore, **the only change that the Hilbert transform performs on a signal is changing its phase**
- The amplitude of the frequency components of the signal do not change by performing the Hilbert-transform

$x(t)$

transform changes cosines into sines, the Hilbert transform of a signal x(t) is orthogonal to x(t)
- Also, since the Hilbert transform introduces a 90· phase shift, carrying it out twice causes a 180· phase shift, which can cause a sign reversal of the original signal

HILBERT TRANSFORM - ITS PROPERTIES

- **Evenness and Oddness**
  - The Hilbert transform of an even signal is odd, and the Hilbert transform of an odd signal is even

- *Proof*
  - If x(t) is even, then X(f) is a real and even function
  - Therefore, -jsgn(f)X(f) is an imaginary and odd function
  - Hence, its inverse Fourier transform $x(t)$ will be odd
  - If x(t) is odd, then *X(*f) is imaginary and odd
  - Thus -jsgn(f)X(f) is real and even
  - Therefore, $x(t)$ is even

HILBERT TRANSFORM - ITS PROPERTIES

- **Sign Reversal**
  - Applying the Hilbert-transform operation to a signal twice causes a sign reversal of the signal, i.e.,

$$\hat{\hat{x}}(t) = -x(t)$$

– *Proof*

$$F[\hat{\hat{x}}(t)] = [-j\text{sgn}(f)]^2 X(f)$$

$$F[\hat{\hat{x}}(t)] = -X(f)$$

- *X(f)* does not contain any impulses at the origin

HILBERT TRANSFORM - ITS PROPERTIES

- **Energy**
  - The energy content of a signal is equal to the energy content of its Hilbert transform

- *Proof*
  - Using Rayleigh's theorem of the Fourier transform,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$E_{\hat{x}} = \int_{-\infty}^{\infty} |\hat{x}(t)|^2 dt = \int_{-\infty}^{\infty} |-j\,\mathrm{sgn}(f)X(f)|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

  - Using the fact that $|-j\mathrm{sgn}(f)|^2 = 1$ except for f = 0, and the fact that X(*f*) does not contain any impulses at the origin completes the proof

HILBERT TRANSFORM - ITS PROPERTIES

- **Orthogonality**
  - The signal *x*(*t*) and its Hilbert transform are orthogonal

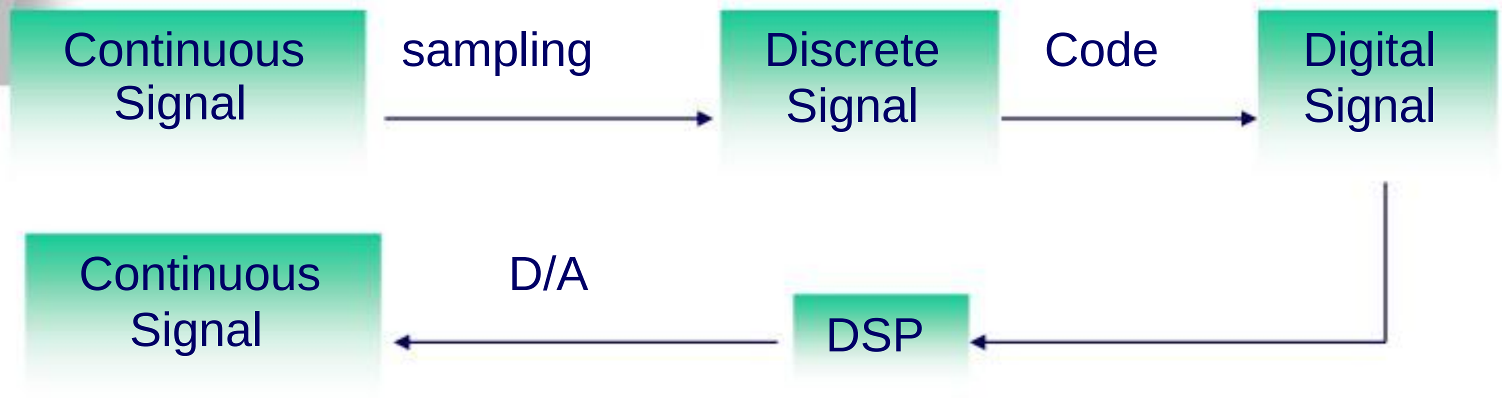
- *Proof*
  - Using Parseval's theorem of the Fourier transform, we obtain

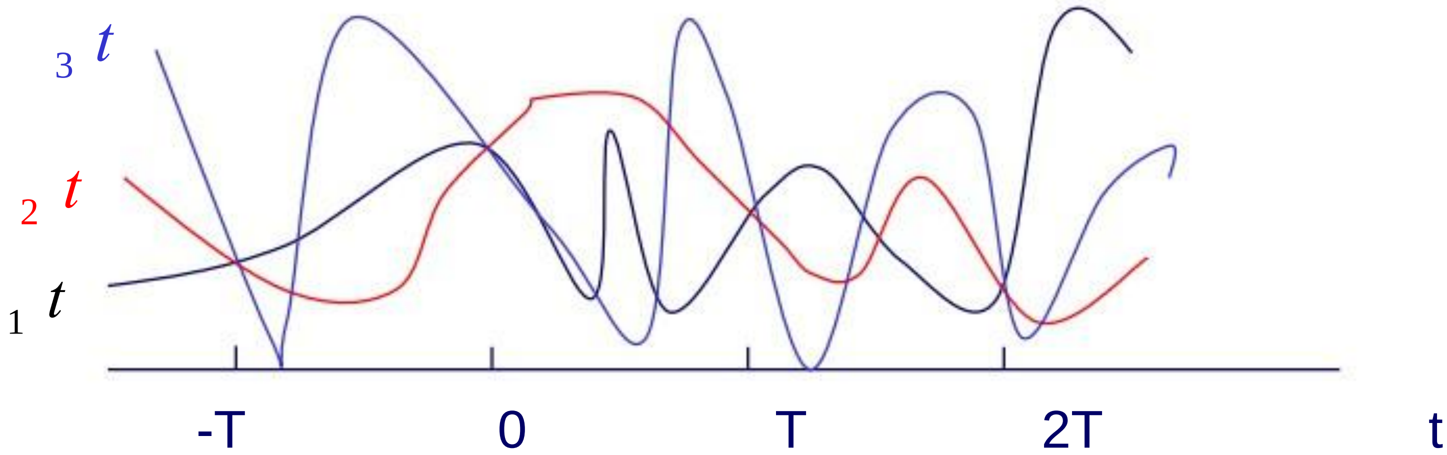
$$\int_{-\infty}^{\infty} x(t)\hat{x}^*(t)\,dt = \int_{-\infty}^{\infty} X(f)[-j\,\mathrm{sgn}(f)X(f)]^*\,df$$

$$= -j\int_{-\infty}^{0} |X(f)|^2\,df + j\int_{0}^{\infty} |X(f)|^2\,df = 0$$

  - In the last step, we have used the fact that X(f) is Hermitian; $|X(f)|^2$ is even

Sampling and reconstruction





$$x_1 \cdot nT \cdot \cdot x_2 \cdot nT \cdot \cdot x_3 \cdot nT \cdot$$

$$x_1 \, t \cdot \cdot x_2 \, t \cdot \cdot x_3 \, t \cdot$$

Sampling: Time Domain

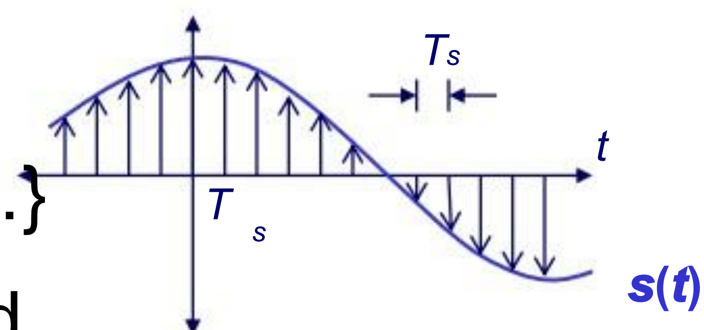
- Many signals originate as continuous-time signals, e.g. conventional music or voice
- By sampling a continuous-time signal at isolated, equally-spaced points in time, we obtain a sequence of numbers

$$s[n] = s(nT_s)$$

$$n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

T_s is the sampling period.

$$s_{sampled}(t) = s(t) \underbrace{\sum_n \delta(t - nT_s)}_{\text{impulse train}}$$



Sampled analog waveform

Sampling: Frequency Domain

- Replicates spectrum of continuous-time signal

  At offsets that are integer multiples of sampling frequency

- Fourier series of impulse train where $w_s = 2 \cdot f_s$

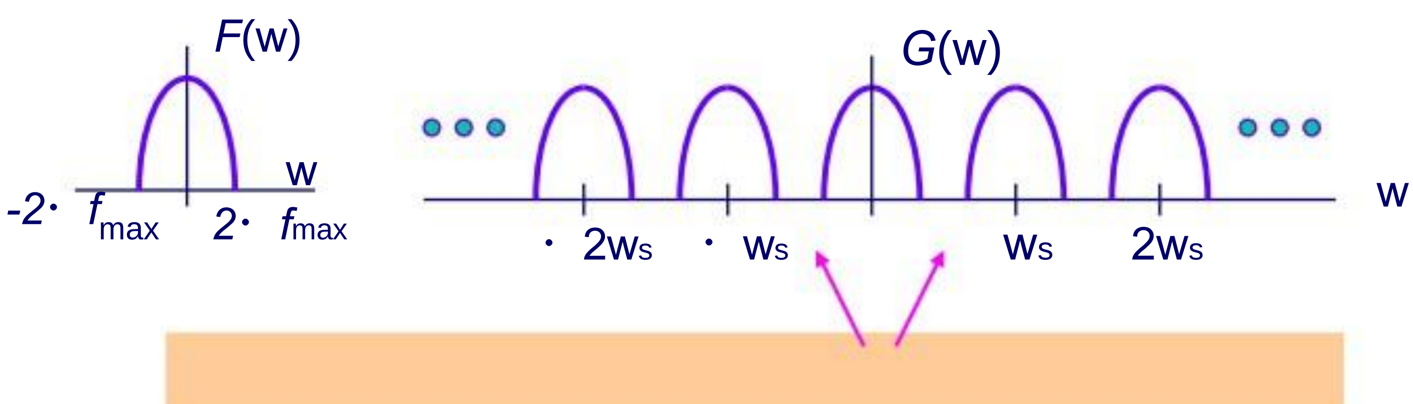
$$\cdot_{T_s}(t) \cdot \sum_{n \cdot \cdot \cdot} \cdot (t \cdot nT_s) \cdot \frac{1}{T_s} \cdot \frac{2}{T_s}\cos(w_s t) \cdot \frac{2}{T_s}\cos(2w_s t) \cdot \ldots$$

$$g(t) \cdot f(t) \cdot_{T_s}(t) \cdot \frac{1}{T_s} \cdot f(t) \cdot 2f(t)\cos(w_s t) \cdot 2f(t)\cos(2w_s t) \cdot \ldots \cdot$$

Modulation by cos(w_s t)

Modulation by cos(2 w_s t)

- Example



Shannon Sampling Theorem

- A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed from its samples $x[n] = x(n\ T_s)$ if the samples are taken at a rate f_s which is greater than $2\ f_{max}$.

  Nyquist rate = $2\ f_{max}$

  Nyquist frequency = $f_s/2$.

- What happens if $f_s = 2f_{max}$?
- Consider a sinusoid sin(2・・ $f_{max}\ t$)

  Use a sampling period of $T_s = 1/f_s = 1/2f_{max}$.

  Sketch: sinusoid with zeros at $t = 0, 1/2f_{max}, 1/\mathrm{f}_{max}$, …

Shannon Sampling Theorem

Assumption

- Continuous-time signal has no frequency content above f_{max}
- Sampling time is exactly the same between any two samples
- Sequence of numbers obtained by sampling is represented in exact precision
- Conversion of sequence to continuous time is ideal

In Practice

Why 44.1 kHz for Audio CDs?

- Sound is audible in 20 Hz to 20 kHz range:

  f_{max} = 20 kHz and the Nyquist rate 2 f_{max} = 40 kHz

- What is the extra 10% of the bandwidth used?

  Rolloff from passband to stopband in the magnitude response of the anti-aliasing filter

- Okay, 44 kHz makes sense. Why 44.1 kHz?

  At the time the choice was made, only recorders capable of storing such high rates were VCRs.

  NTSC: 490 lines/frame, 3 samples/line, 30 frames/s = 44100 samples/s

  PAL: 588 lines/frame, 3 samples/line, 25 frames/s = 44100 samples/s

Sampling

- As sampling rate increases, sampled waveform looks more and more like the original
- Many applications (e.g. communication systems) care more about frequency content in the waveform and not its shape
- Zero crossings: frequency content of a sinusoid
  - Distance between two zero crossings: one half period.
  - With the sampling theorem satisfied, sampled sinusoid crosses zero at the right times even though its waveform shape may be difficult to recognize

Aliasing

- Analog sinusoid

  $x(t) = A \cos(2\cdot\ f_0 t + \cdot\ )$

- Sample at $T_s = 1/f_s$

  $x[n] = x(T_s\, n) =$

  $A \cos(2\cdot\ \cdot\ f_0\, T_s\, n + \cdot\ )$

- Keeping the sampling period same, sample

  $y(t) = A \cos(2\cdot\ (f_0 + l f_s) t\ \ + \cdot\ )$

  where l is an integer

$y[n] = y(T_s\, n)$

$= A \cos(2\cdot\ (f_0 + l f_s) T_s n + \cdot\ )$

$= A \cos(2\cdot\ f_0 T_s n + 2\cdot\ \cdot\ l f_s T_s n + \cdot\ )$

$= A \cos(2\cdot\ f_0 T_s n + 2\cdot\ \cdot\ l\, n + \cdot\ )$

$= A \cos(2\cdot\ f_0 T_s n + \cdot\ )$

$= x[n]$

Here, $f_s T_s = 1$

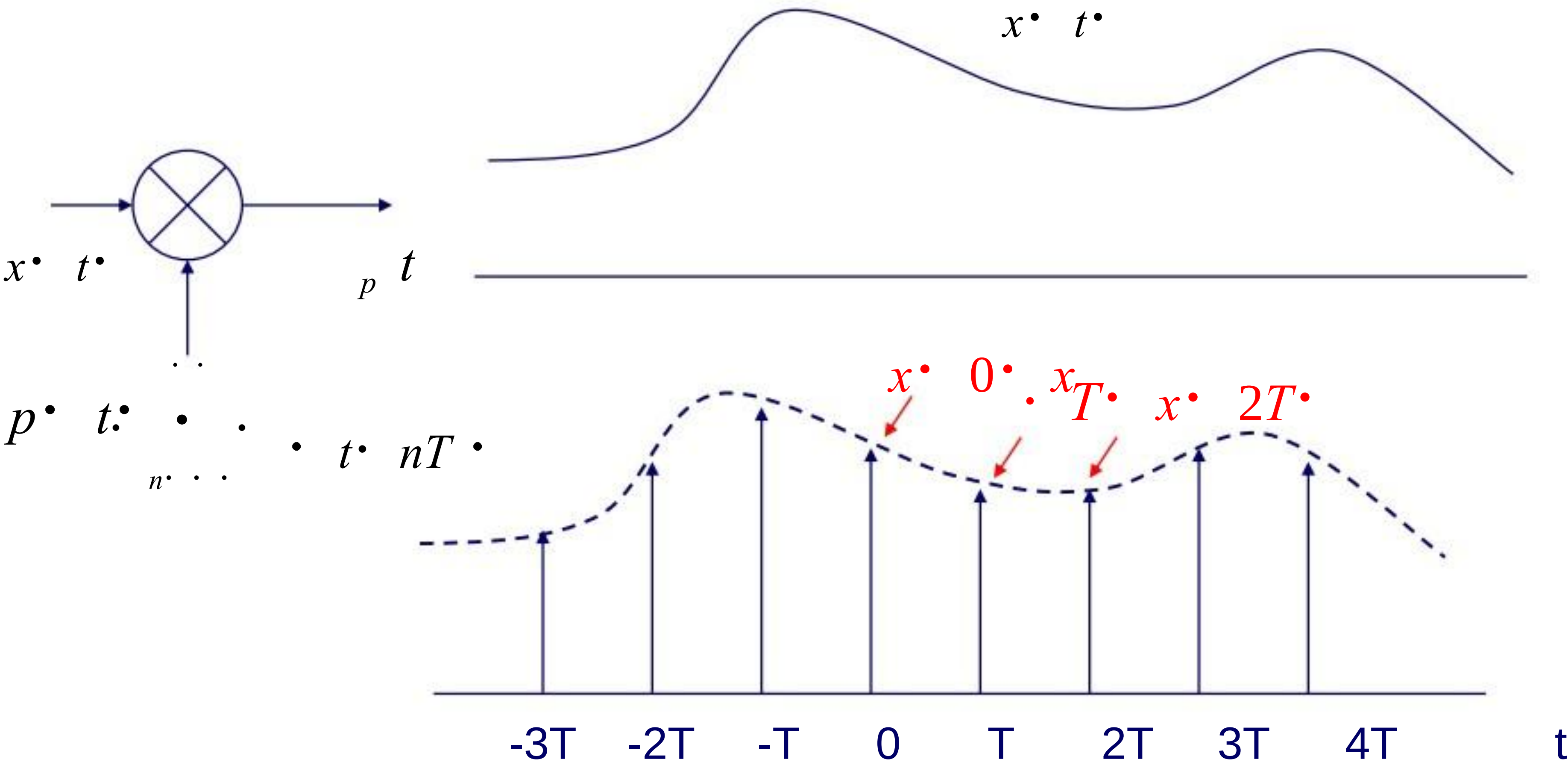
Since l is an integer,

$\cos(x + 2\cdot\ l) = \cos(x)$

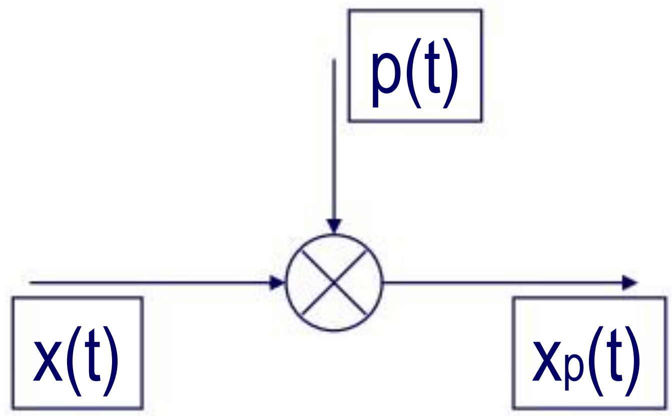
- $y[n]$ indistinguishable from $x[n]$

Frequencies $f_0 + l\, f_s$ for $l \cdot\ \ 0$ are aliases of frequency f_0

Impulse-Train Sampling



Sampling



$$x_p\, t) \cdot x\, t)p(t)$$

$$X_p(jw) \cdot \frac{1}{2\cdot}[X(jw) * P(jw)]$$

$$where\ \ p(t) \cdot \cdot \quad {}_T(t) \cdot \quad \bullet \quad \cdot \quad (t \cdot \ nT)$$

Time domain:

$$x_p(t) = x(t) \cdot \delta_T(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

Frequency domain:

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

$$p(t) \overset{F.S.}{\longleftrightarrow} a_k = \frac{1}{T} \quad (Periodic \quad signal)$$

$$p(t) \overset{F}{\longleftrightarrow} P(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_s) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$x_p(t) \overset{F}{\longleftrightarrow} X_p(j\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

$$X_p(j\omega) = X(j\omega) * P(j\omega)$$

$$\omega_s > 2\omega_M$$

Sampling Theorem:

Let $x(t)$ be a band-limited signal with $X(j\omega) = 0, \ |\omega| > \omega_M$

then $x(t)$ is uniquely determined by its samples $x(nT), n = 0, \pm 1, \pm 2, \cdots$

if $\omega_s > 2\omega_M$ where $\omega_s = \frac{2\pi}{T}$

$$2\omega_M : Nyquisy\ Rate$$

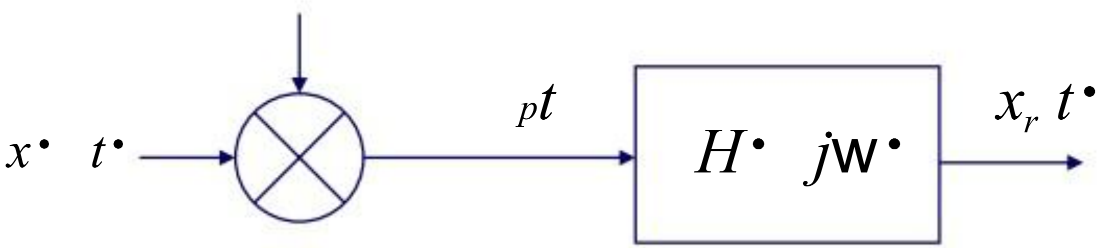
(Minimum distortionless sampling frequency)

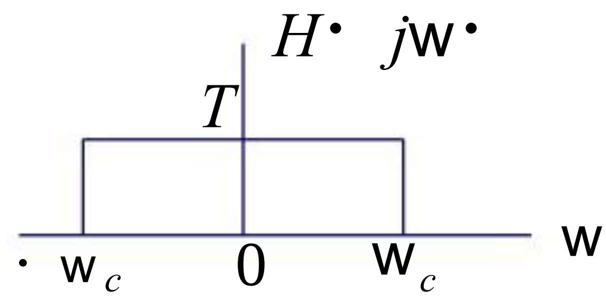
$$\omega_M : Nyquist Frequency$$

(Maximum distortionless sampled signal frequency)

The reconstruction of the signal

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$$



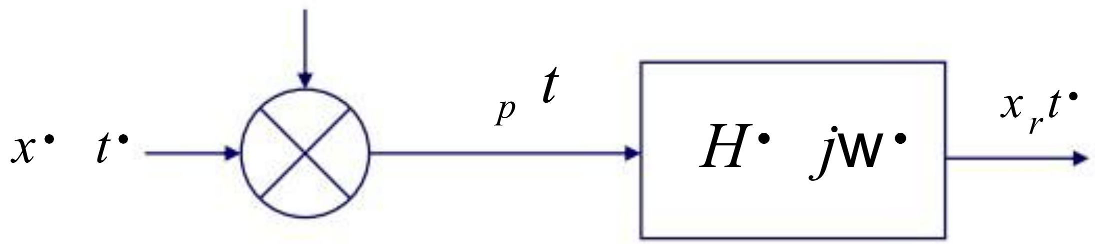


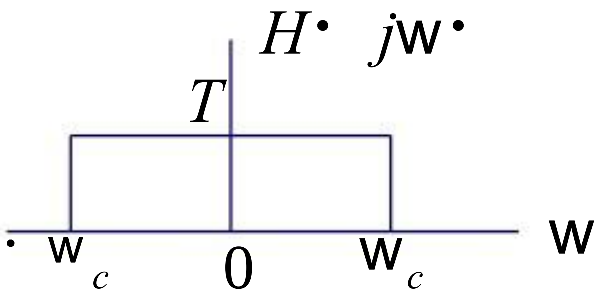
$\omega_M < \omega_c < \omega_s - \omega_M$

$$x(t)=\sum_{n=-\infty}^{\infty}x(nT)\,S_a(\omega_H(t-nT))$$

Natural Sampling

$$p(t) = \sum_{n=-\infty}^{\infty} P_{\tau}(t - nT)$$





$$\omega_M < \omega_c < \omega_s - \omega_M$$

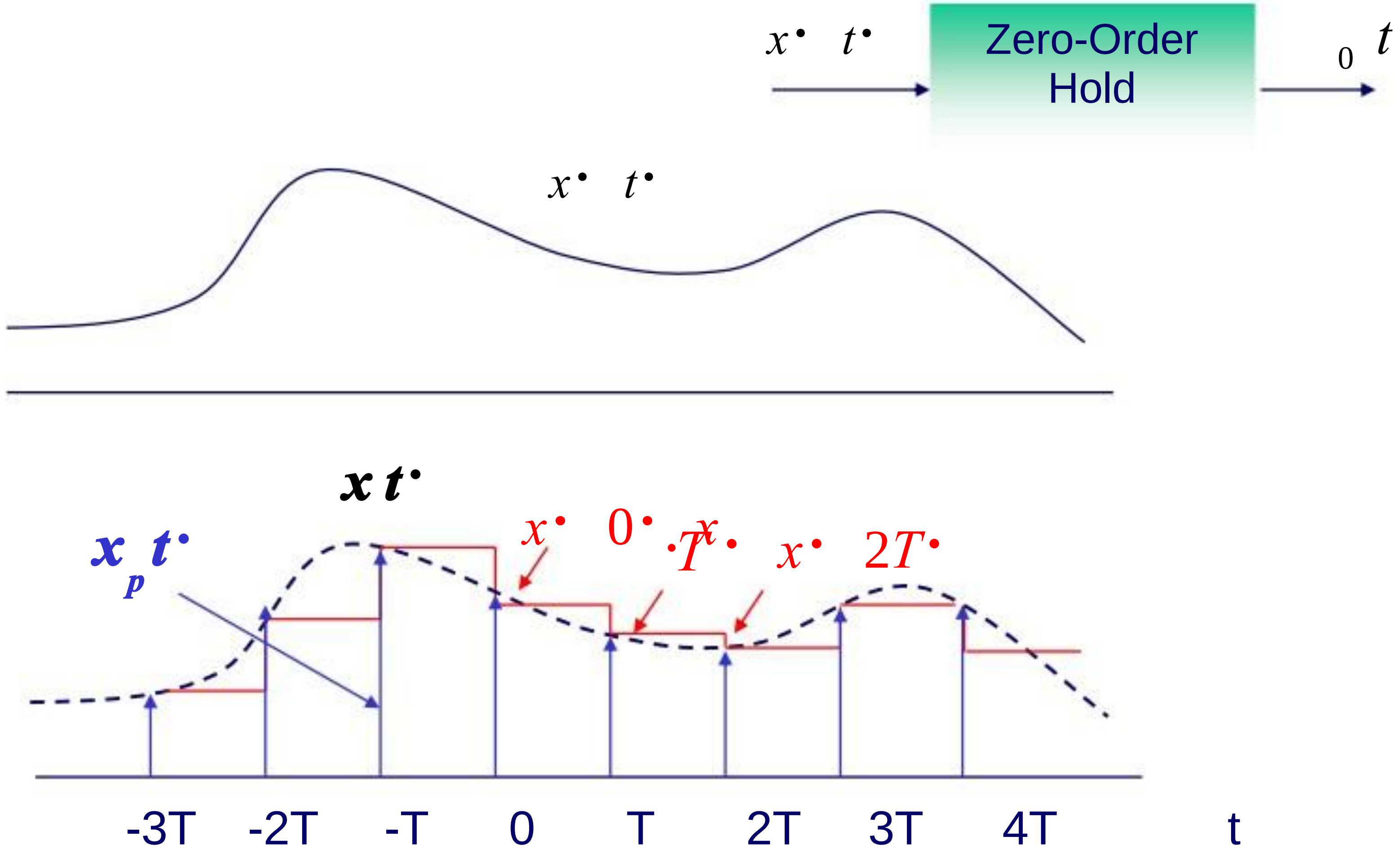
$$x_r(t) = \left[x(t) \cdot \sum_{n=-\infty}^{\infty} P_{\tau}(t - nT) \right] * h(t)$$

Difficult:

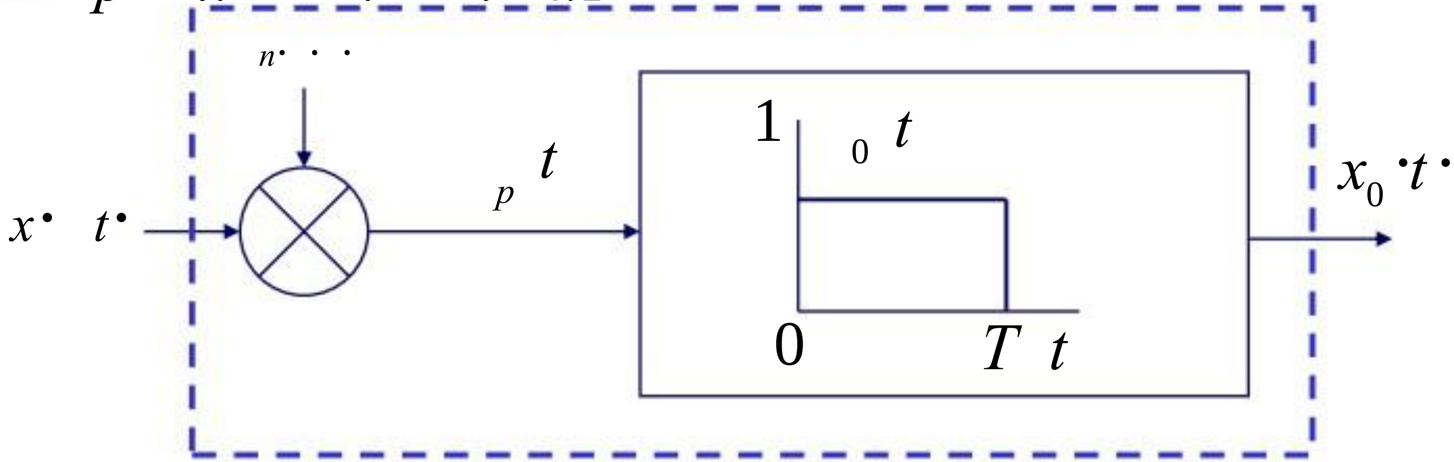
1 ILPF is unpractical;

2 narrow, large-amplitude pulses are difficult to generate and transmit.

Sampling with a Zero-Order Hold



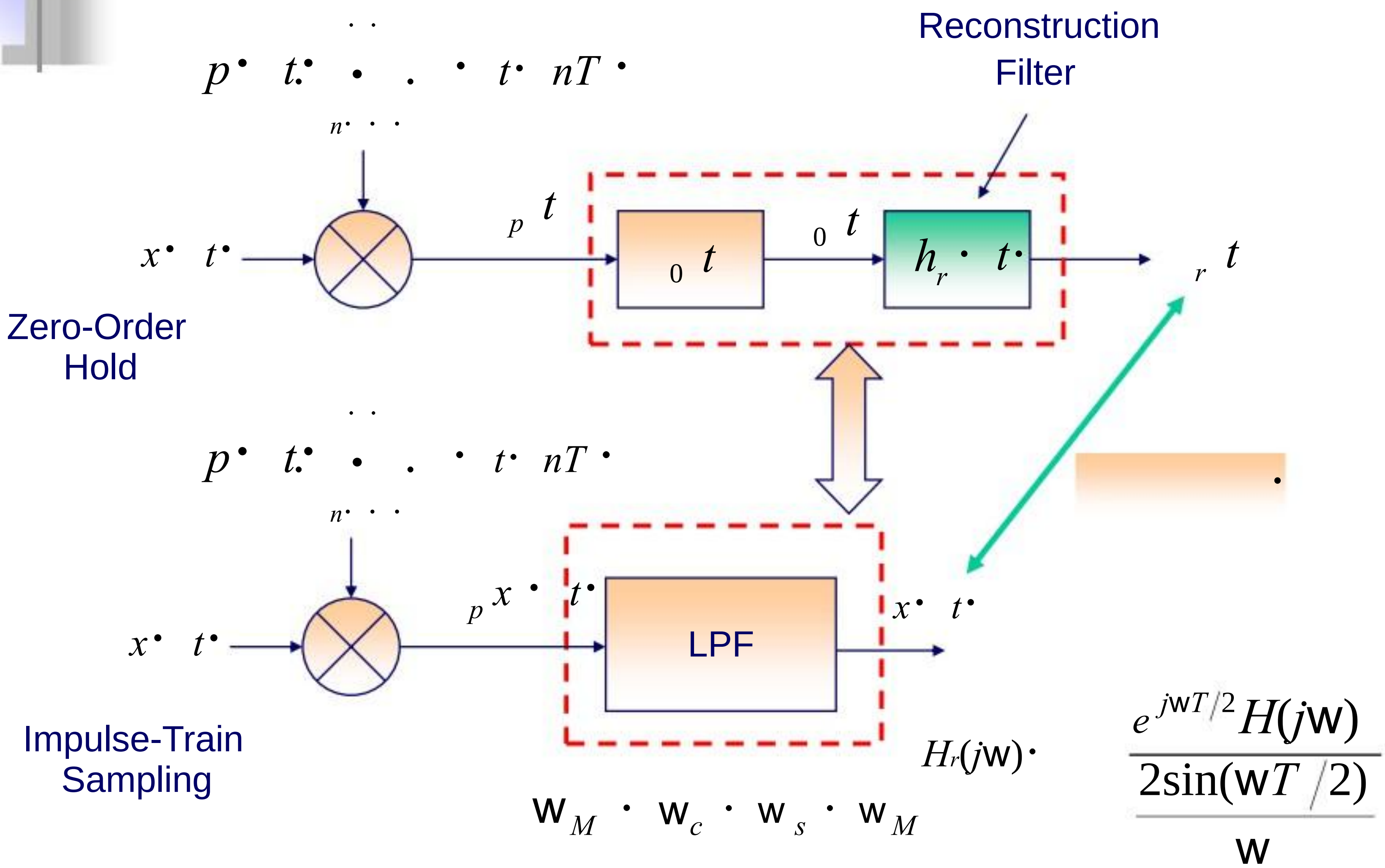
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Zero-Order Hold

$$x_0(t) = x_p(t) * h_0(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h_0(t - nT)$$

$$H_o(j\omega) = e^{-j\omega T/2}\left(\frac{2\sin(\omega T/2)}{\omega}\right)$$

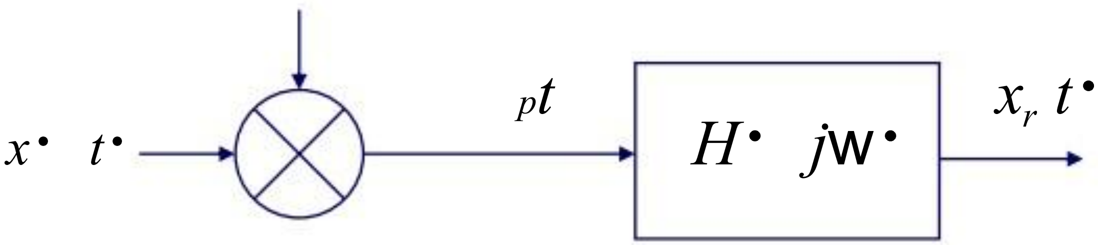

Reconstruction
Filter
Zero-Order
Hold
LPF
Impulse-Train
Sampling
$H_r(jw)$
$\frac{e^{jwT/2}H(jw)}{2\sin(wT/2)}$
w

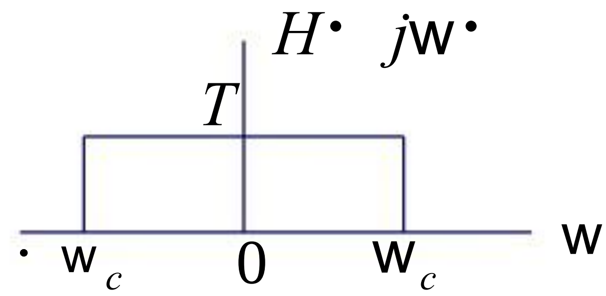
$$H_r(j\omega) = \frac{e^{j\omega T/2} H(j\omega)}{\dfrac{2\sin(\omega T/2)}{\omega}}$$

Reconstruction

Band-limited interpolation

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$





$$\omega_M < \omega_c < \omega_s - \omega_M$$

$$x_r(t) = x_p(t) * h(t) \qquad h(t) = \frac{T \sin(\omega_c t)}{\pi t}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT) * h(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\,h(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\,\frac{T \sin[\omega_c (t - nT)]}{\pi (t - nT)}$$

Original CT Signal

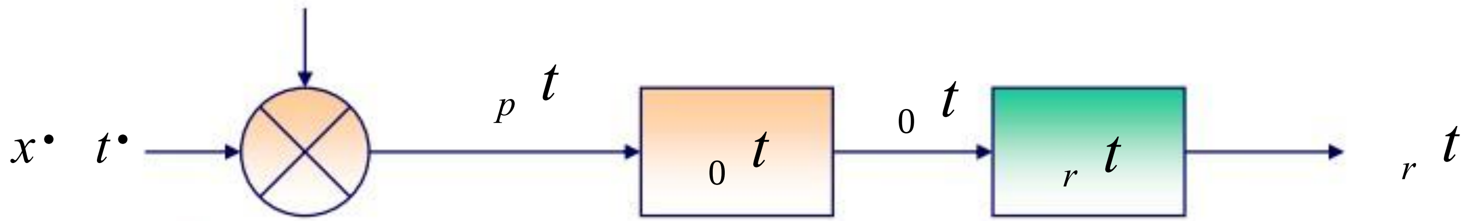
After sampling

After passing LPF

The LPF smoothes out shape and fill in the gaps

Zero-order hold

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Original CT Signal

After sampling

After passing
zero-order hold

$$H_o(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right] H(j\omega)$$

Zero-Order Hold

$$H_r(j\omega) = \frac{e^{j\omega T/2} H(j\omega)}{\frac{2\sin(\omega T/2)}{\omega}}$$

Zero-Order Hold
Recover Filter

[illegible]

Sampling theory

x *

= =

$f(x)$ $F(s)$

Spectrum is replicated an infinite number of times

Reconstruction theory

$*$

sinc

x

$=$ $=$

$f(x)$ $F(s)$

Sampling at the Nyquist rate

x   *

=   =

$f(x)$ $F(s)$

Reconstruction at the Nyquist rate

*

x

=

=

$f(x)$

$F(s)$

Sampling below the Nyquist rate

x

*

=

=

$f(x)$

$F(s)$

Reconstruction below the Nyquist rate

*

x

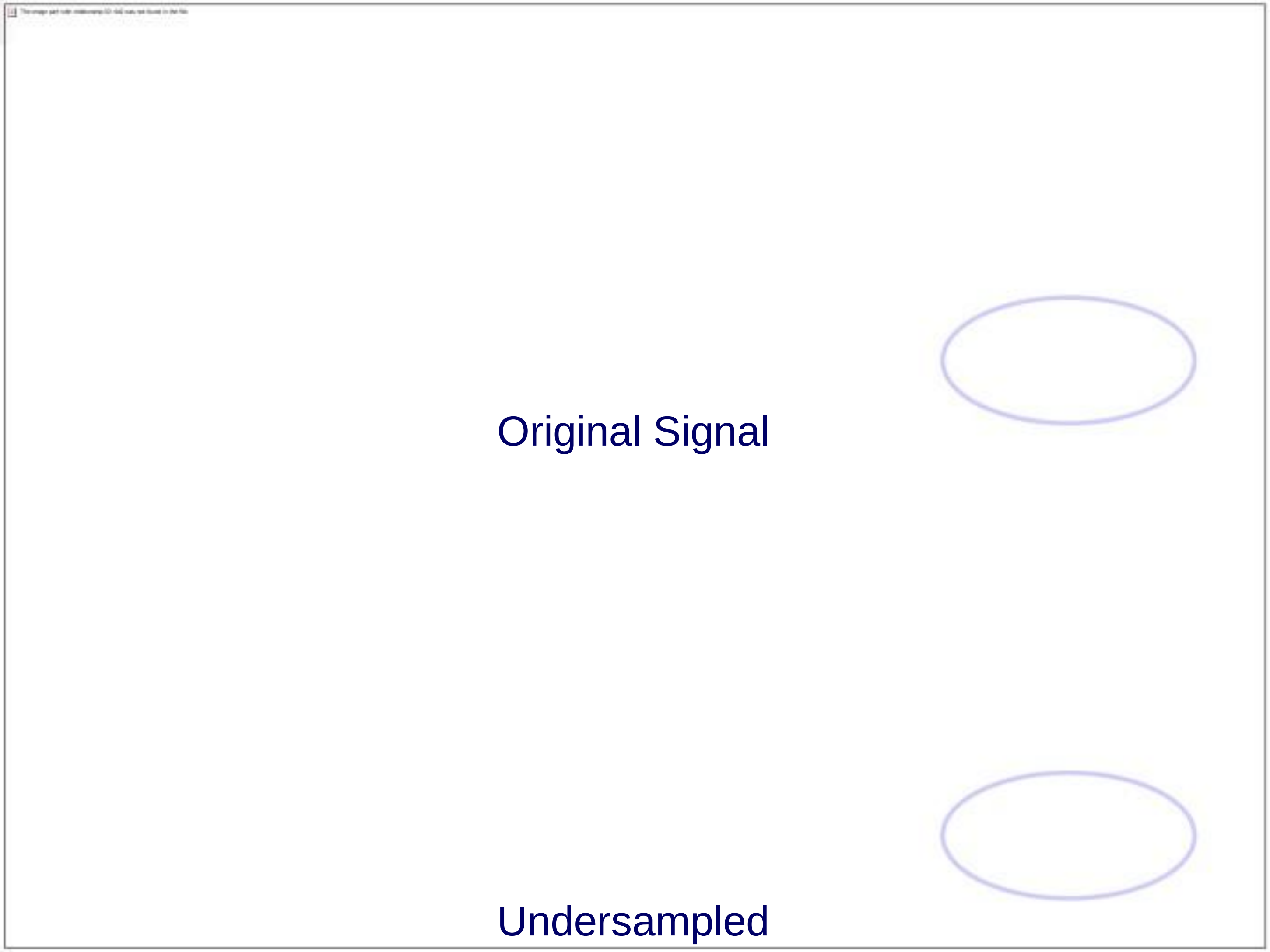
=

=

$f(x)$

$F(s)$

Reconstruction error



Reconstruction with a triangle function

*

x

=

=

f(x)

F(s)

Reconstruction error

Original Signal

Triangle
Reconstruction

Reconstruction with a rectangle function

*

x

=

=

$f(x)$

$F(s)$

Reconstruction error

Original Signal

Rectangle Reconstruction

Sampling a rectangle

x *

= =

$f(x)$ $F(s)$

Reconstructing a rectangle (jaggies)

*

x

=

=

$f(x)$ $F(s)$

Sampling and reconstruction

Aliasing is caused by

- Sampling below the Nyquist rate,
- Improper reconstruction, or
- Both

We can distinguish between

- Aliasing of fundamentals (demo)
- Aliasing of harmonics (jaggies)

Time-Domain System Analysis

Impulse Response

Let a system be described by

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = x(t)$$

and let the excitation be a unit impulse at time $t = 0$. Then the

zero-state response y is the impulse response h.

$$a_2 h''(t) + a_1 h'(t) + a_0 h(t) = \delta(t)$$

Since the impulse occurs at time $t = 0$ and nothing has excited the system before that time, the impulse response before time $t = 0$ is zero (because this is a causal system). After time $t = 0$ the impulse has occurred and gone away. Therefore there is no longer an excitation and the impulse response is the homogeneous solution of the differential equation.

Impulse Response

$$_{2}\, h''(t) + a_1 h'(t) + a_0 h(t) = \delta(t)$$

What happens at time, $t = 0$? The equation must be satisfied at all times. So the left side of the equation must be a unit impulse. We already know that the left side is zero before time $t = 0$ because the system has never been excited. We know that the left side is zero after time $t = 0$ because it is the solution of the homogeneous equation whose right side is zero. These two facts are both consistent with an impulse. The impulse response might have in it an impulse or derivatives of an impulse since all of these occur only at time, $t = 0$. What the impulse response does have in it depends on the form of the differential equation.

Impulse Response

Continuous-time LTI systems are described by differential equations of the general form,

$$(\ )(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t)$$

$$(\ )(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_1 x'(t) + b_0 x(t)$$

For all times, $t < 0$:

If the excitation $x(t)$ is an impulse, then for all time $t < 0$ it is zero. The response $y(t)$ is zero before time $t = 0$ because there has never been an excitation before that time.

Impulse Response

For all time $t > 0$:

The excitation is zero. The response is the homogeneous solution of the differential equation.

At time $t = 0$:

The excitation is an impulse. In general it would be possible for the response to contain an impulse plus derivatives of an impulse because these all occur at time $t = 0$ and are zero before and after that time. Whether or not the response contains an impulse or derivatives of an impulse at time $t = 0$ depends on the form of the differential equation

$$^{(\ )}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t)$$

$$^{(\ )}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_1 x'(t) + b_0 x(t)$$

Impulse Response

$$^{(\ )}(t)+a_{n-1}y^{(n-1)}(t)+\cdot\ \ +a_1\quad y'(t)+a_0\,y(t)$$

$$^{(\ )}(t)+b_{m-1}x^{(m-1)}(t)+\cdot\ \ +b_1\quad x'(t)+b_0\,x(t)$$

Case 1: $m < n$

If the response contained an impulse at time $t = 0$ then the nth derivative of the response would contain the nth derivative of an impulse. Since the excitation contains only the mth derivative of an impulse and $m < n$, the differential equation cannot be satisfied at time $t = 0$. Therefore the response cannot contain an impulse or any derivatives of an impulse.

Impulse Response

$$( )(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_1 y'(t)+a_0 y(t)$$

$$( )(t)+b_{m-1}x^{(m-1)}(t)+\cdots+b_1 x'(t)+b_0 x(t)$$

Case 2: $m = n$

In this case the highest derivative of the excitation and response are the same and the response could contain an impulse at time $t = 0$ but no derivatives of an impulse.

Case 3: $m > n$

In this case, the response could contain an impulse at time $t = 0$ plus derivatives of an impulse up to the $(m - n)$th derivative.

Case 3 is rare in the analysis of practical systems.

Impulse Response

Example

Let a system be described by $y'(t) + 3y(t) = x(t)$. If the excitation x is an impulse we have $h'(t) + 3h(t) = \delta(t)$. We know that $h(t) = 0$ for $t < 0$ and that $h(t)$ is the homogeneous solution for $t > 0$ which is $h(t)=Ke^{-3t}$. There are more derivatives of y than of x. Therefore the impulse response cannot contain an impulse. So the impulse response is $h(t) = Ke^{-3t}\,u(t)$.

Impulse Response

Example

To find the constant K integrate $\mathrm{h}'(t)+3\,\mathrm{h}(t)=\delta(t)$ over the infinitesimal range 0^- to 0^+.

$$\int_{0^-}^{0^+}\mathrm{h}'(t)dt+3\int_{0^-}^{0^+}\mathrm{h}(t)=\int_{0^-}^{0^+}\delta(t)$$

$$\underbrace{\mathrm{h}(0^+)}_{=K}-\underbrace{\mathrm{h}(0^-)}_{=0}+3\int_{0^-}^{0^+}Ke^{-3t}\,\mathrm{u}(t)dt=\underbrace{\mathrm{u}(0^+)}_{=1}-\underbrace{\mathrm{u}(0^-)}_{=0}$$

$$K+3K\left[\frac{e^{-3t}}{-3}\right]_0^{0^+}=K+3K\underbrace{\left[(-1/3)-(-1/3)\right]}_{=0}=1$$

$$K=1\Rightarrow\mathrm{h}(t)=e^{-3t}\,\mathrm{u}(t)$$

Impulse Response

Example

To check the solution, put it into the differential equation to see whether it is satisfied.

$$\frac{d}{dt}\left(e^{-3t}\,\mathrm{u}(t)\right)+3e^{-3t}\,\mathrm{u}(t)=\delta(t)$$

$$e^{-3t}\delta(t)-3e^{-3t}\,\mathrm{u}(t)+3e^{-3t}\,\mathrm{u}(t)=\delta(t)$$

$$\underbrace{e^{-3t}\delta(t)}_{=e^{0}\delta(t)=\delta(t)}=\delta(t)\Rightarrow\delta(t)=\delta(t)\quad \text{Check.}$$

Impulse Response

Example

Let a system be described by $4\,\mathrm{y}'(t)+3\,\mathrm{y}(t)=\mathrm{x}'(t)$. The homogeneous solution is $\mathrm{y}_{\mathrm{h}}(t)=Ke^{-3t/4}$ and that is the form of the impulse response for $t>0$. The number of y derivatives and the number of x derivatives are the same. Therefore the impulse response has an impulse in it and its form is $\mathrm{h}(t)=Ke^{-3t/4}\,\mathrm{u}(t)+K_{\delta}\delta(t)$. Integrate between 0^- and 0^+.

$$4\int_{0^-}^{0^+}\mathrm{h}'(t)\,dt+3\int_{0^-}^{0^+}\mathrm{h}(t)\,dt=\int_{0^-}^{0^+}\delta'(t)\,dt$$

$$\left\{\begin{array}{l} 4\left[\underbrace{\mathrm{h}(0^+)}_{=K}-\underbrace{\mathrm{h}(0^-)}_{=0}+K_{\delta}\left(\underbrace{\delta(0^+)}_{=0}-\underbrace{\delta(0^-)}_{=0}\right)\right] \\ +3\underbrace{\int_{0^-}^{0^+}Ke^{-3t/4}\,\mathrm{u}(t)\,dt}_{=0}+3K_{\delta}\left[\underbrace{\mathrm{u}(0^+)}_{=1}-\underbrace{\mathrm{u}(0^-)}_{=0}\right] \end{array}\right\}=\underbrace{\delta(0^+)}_{=0}-\underbrace{\delta(0^-)}_{=0}$$

Impulse Response

Example

$$4K + 3K_\delta = 0$$

Now integrate again over the same infinitesimal interval.

$$4\int_{0}^{0^+}\int_{-\infty}^{t} h'(\lambda)\,d\lambda\,dt + 3\int_{0^-}^{0^+}\int_{-\infty}^{t} Ke^{-3\lambda/4}\,u(\lambda)\,d\lambda\,dt + 3\int_{0^-}^{0^+}\int_{-\infty}^{t} K_\delta\delta(\lambda)\,d\lambda\,dt = \int_{0^-}^{0^+}\int_{-\infty}^{t} \delta'(\lambda)\,d\lambda\,dt$$

$$4\underbrace{\int_{0^-}^{0^+} h(t)\,dt}_{=K_\delta} - 4K\underbrace{\int_{0^-}^{0^+}\left(1-e^{-3t/4}\right)u(t)\,dt}_{=0} + 3K_\delta\underbrace{\int_{0^-}^{0^+} u(t)\,dt}_{=0} = \underbrace{\int_{0^-}^{0^+} \delta(t)\,dt}_{=1}$$

$$4K_\delta = 1 \rightarrow K_\delta = 1/4 \rightarrow 4K + 3/4 = 0 \rightarrow K = -3/16$$

$$h(t) = (-3/16)e^{-3t/4}\,u(t) + (1/4)\delta(t)$$

Impulse Response

Example $h(t) = (-3/16)e^{-3t/4}\,u(t) + (1/4)\delta(t)$

The original differential equation is $4h'(t) + 3h(t) = \delta'(t)$.

Substituting the solution we get

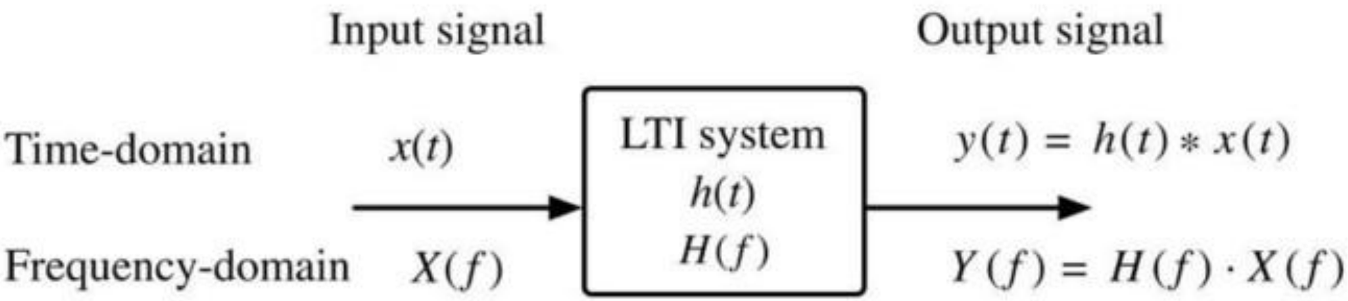
$$\left\{ 4\frac{d}{dt}\left[(-3/16)e^{-3t/4}\,u(t) + (1/4)\delta(t)\right] + 3\left[\quad\right] \right\} = \delta'(t)$$

$$\left\{ 4\left[(-3/16)e^{-3t/4}\delta(t) + (9/64)e^{-3t/4}\,u(t) + (1/4)\delta'(t)\right] + 3\left[\quad\right] \right\} = \delta'(t)$$

$$-(3/4)e^{-3t/4}\delta(t) + (9/16)e^{-3t/4}\,u(t) + \delta'(t) - (9/16)e^{-3t/4}\,u(t) + (3/4)\delta(t) = \delta'(t)$$

$\delta'(t) = \delta'(t)$ Check.

Signal Transmission Through a Linear System



H(f): Transfer function/frequency response

Signal transmission through a linear time-invariant system.

Distortionless transmission:

a signal to pass without distortion

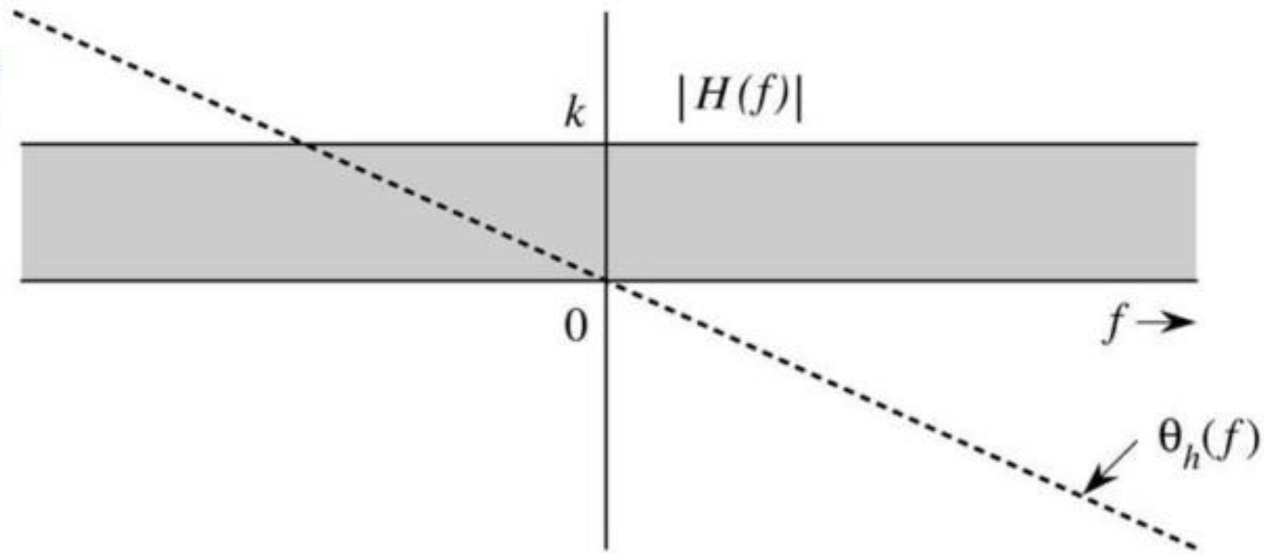
delayed ouput retains the waveform

$$y(t) = k \cdot x(t - t_d)$$

$$Y(f) = k \cdot X(f) e^{-j2\pi f t_d}$$

$$Y(f) = X(f) H(f)$$

$$H(f) = k e^{-j2\pi f t_d}$$



Linear time invariant system frequency response for distortionless transmission.

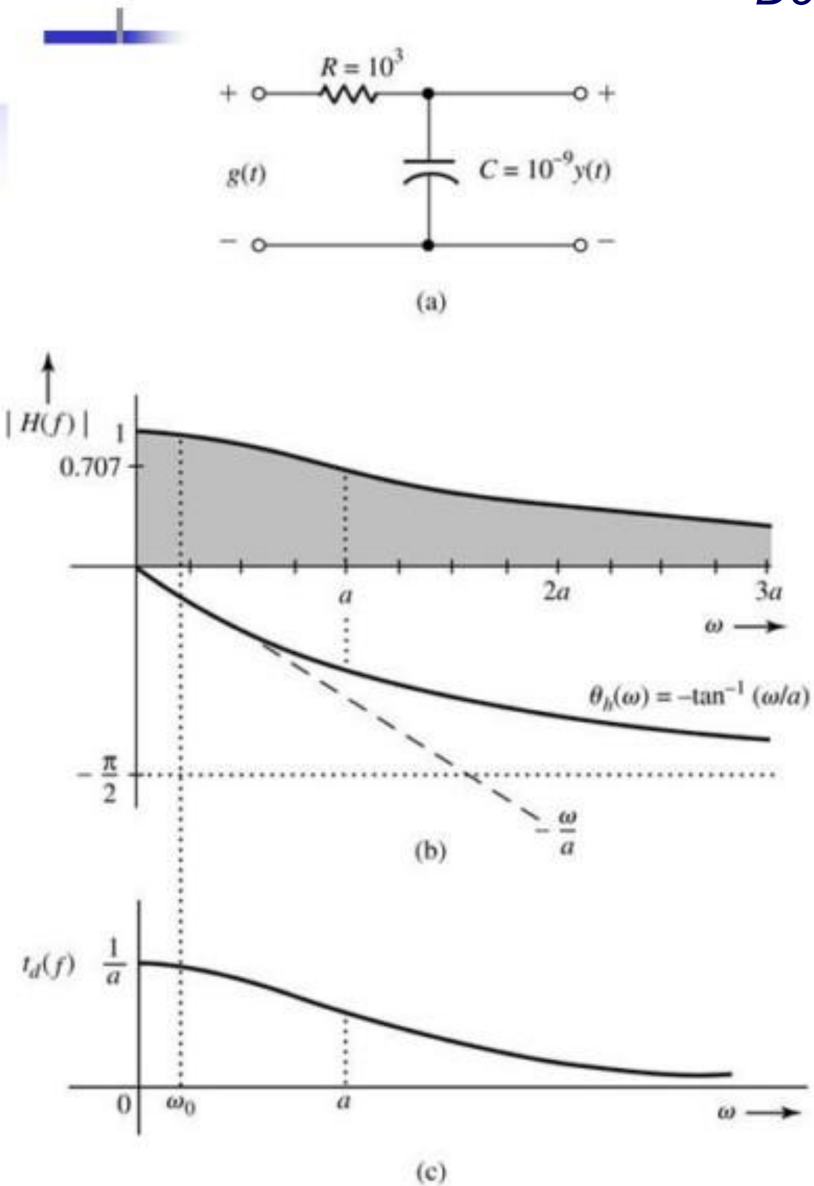
Determine the transfer function H(f), $\theta_h(f)$ *and td(f).*

$$H(f) = \frac{1/j2\pi fC}{R + 1/j2\pi fC} = \frac{1}{1 + j2\pi fRC} = \frac{a}{a + j2\pi f} \qquad a = \frac{1}{RC}$$

$$|H(f)| = \frac{a}{\sqrt{a^2 + (2\pi f)^2}} \approx 1 \quad \text{when } |2\pi f| \ll a$$

$$\theta_h(f) = -\tan^{-1}\frac{2\pi f}{a} \approx -\frac{2\pi f}{a} \quad \text{when } |2\pi f| \ll a$$

$$t_d(f) = -\frac{d\theta_h}{d(2\pi f)} = \frac{a}{a^2 + (2\pi f)^2} \approx \frac{1}{a} \quad \text{when } |2\pi f| \ll a$$



What is the requirement on the bandwidth of g(t) if amplitude variation within 2% and time delay variation within 5% are tolerable?

$$|H(f)| = \frac{a}{\sqrt{a^2 + (2\pi f)^2}} \geq 0.98 \qquad f = 32.31\,kHz$$

$$t_d(f) = \frac{a}{a^2 + (2\pi f)^2} \geq \frac{0.95}{a} \qquad f = 36.51\,kHz$$

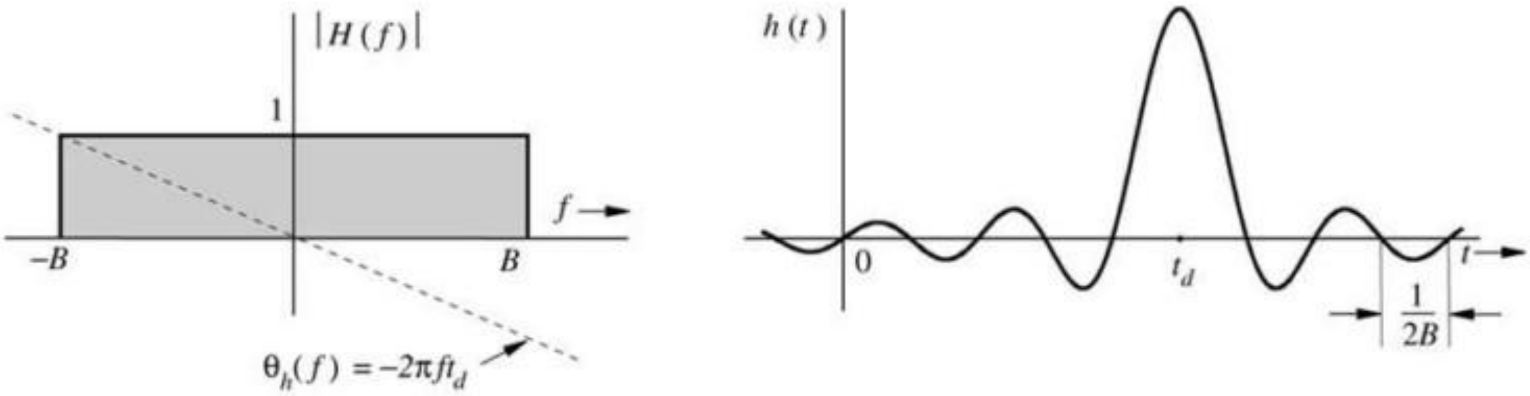
(a) Simple *RC* filter. (b) Its frequency response and time delay.

Ideal filters: allow distortionless transmission of a certain band of frequencies and suppress all the remaining frequencies.

$$y(t) = Kx(t - t_d)$$

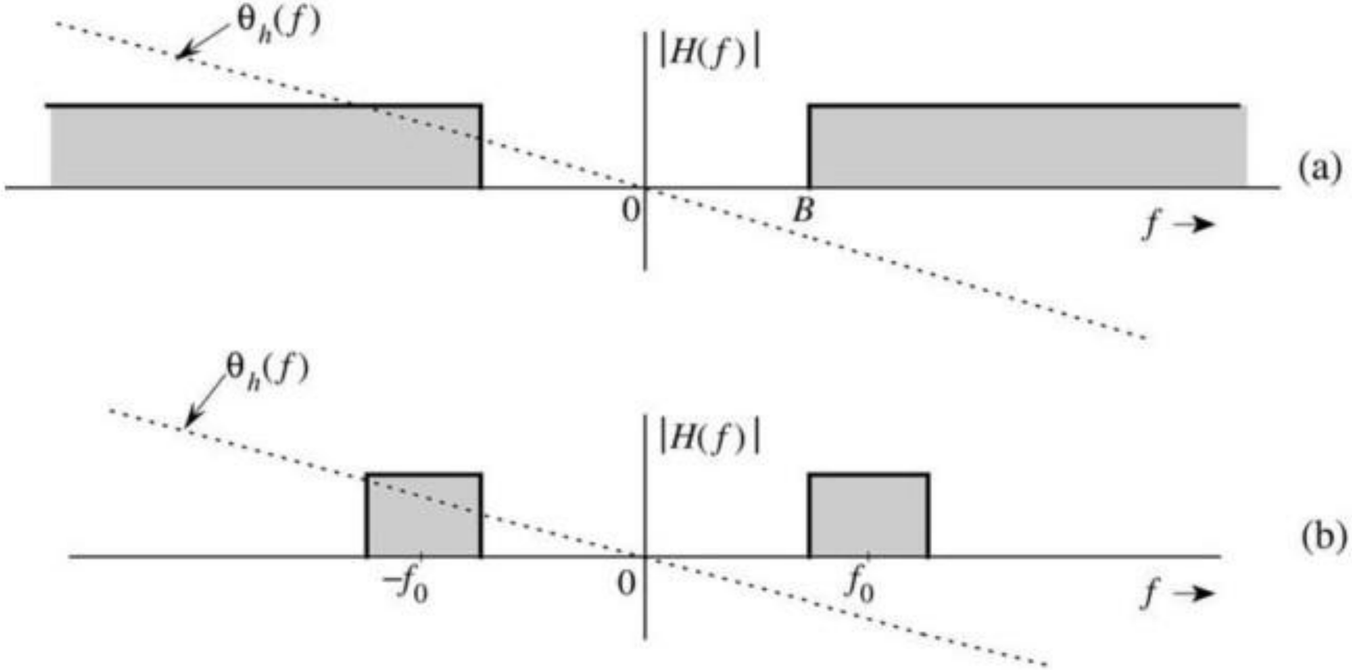
$$H(f) = K\Pi\left(\frac{f}{2B}\right)e^{-j2\pi f t_d}$$

$$h(t) = \mathcal{F}^{-1}\left[K\Pi\left(\frac{f}{2B}\right)e^{-j2\pi f t_d}\right] = 2BK\,\mathrm{sinc}[2B(t - t_d)]$$



(a) (b)

(a) Ideal low-pass filter frequency response and (b) its impulse response.



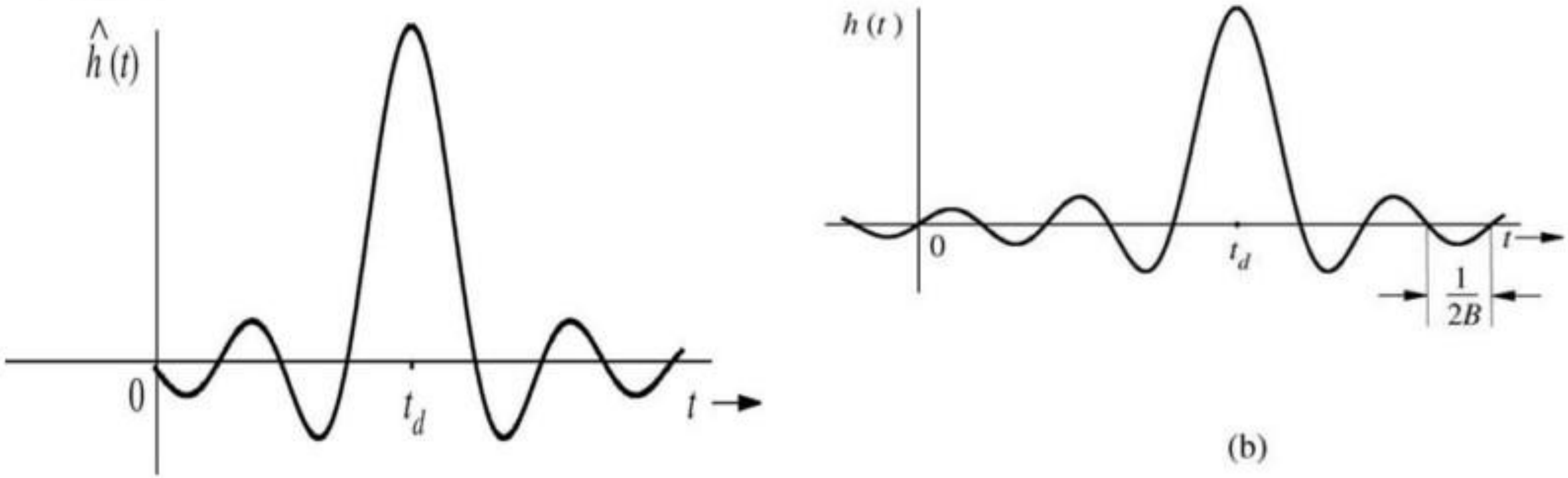
Ideal high-pass and bandpass filter frequency responses.

Paley-Wiener criterion

$$\int_{-\infty}^{\infty} \frac{\left|\ln|H(f)|\right|}{1+(2\pi f)^2}\, df < \infty$$

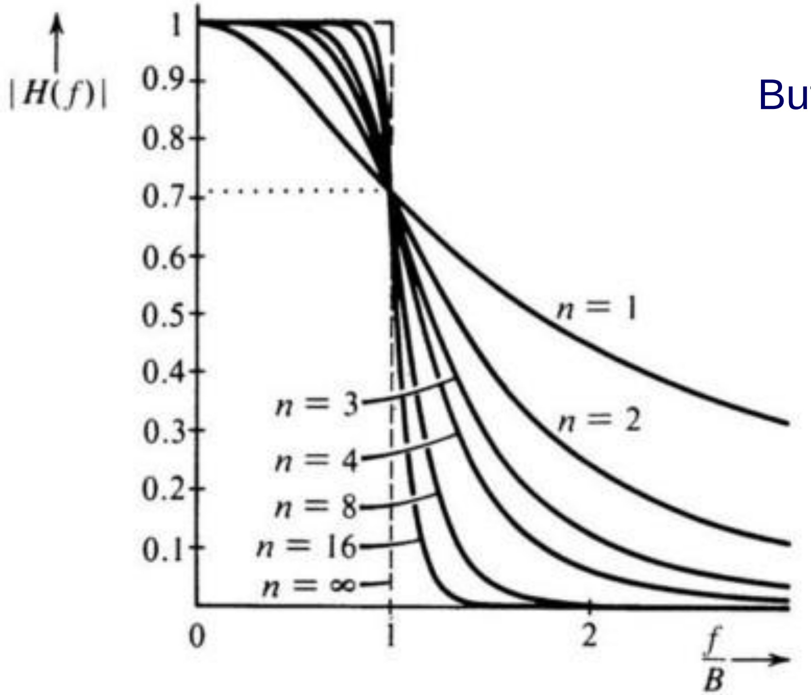
For a physically realizable system h(t) must be causal

$h(t)=0 \quad \text{for } t<0$



Approximate realization of an ideal low-pass filter by truncating its impulse response.

$$\hat{h}(t) = h(t)u(t)$$

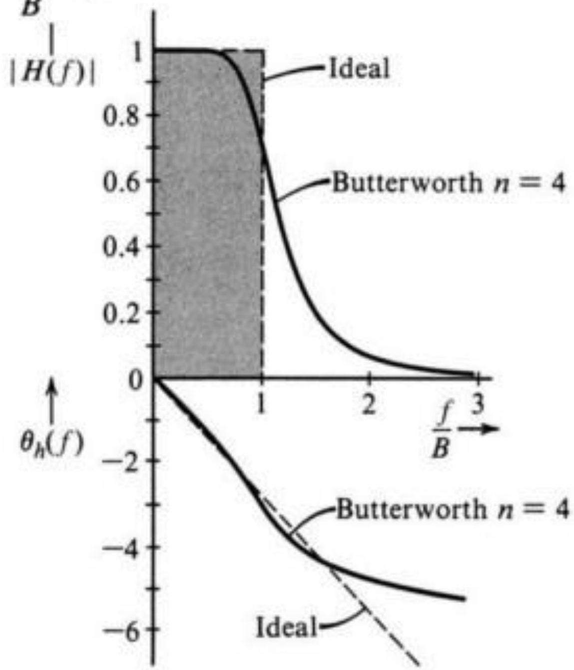


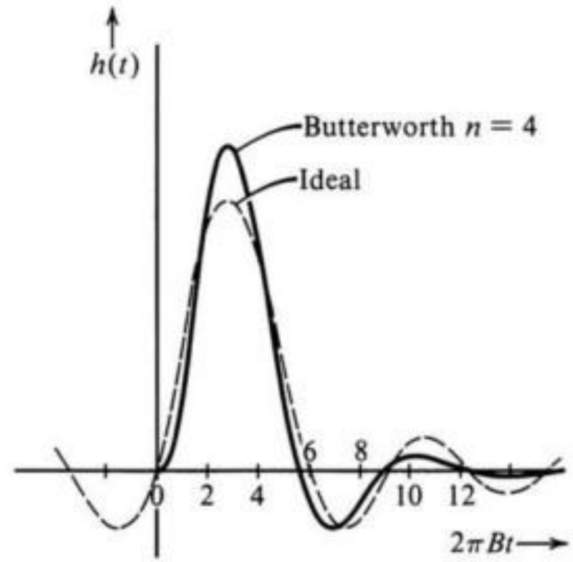
Butterworth filter characteristics.

$$|H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{B}\right)^{2n}}}$$

The half-power bandwidth

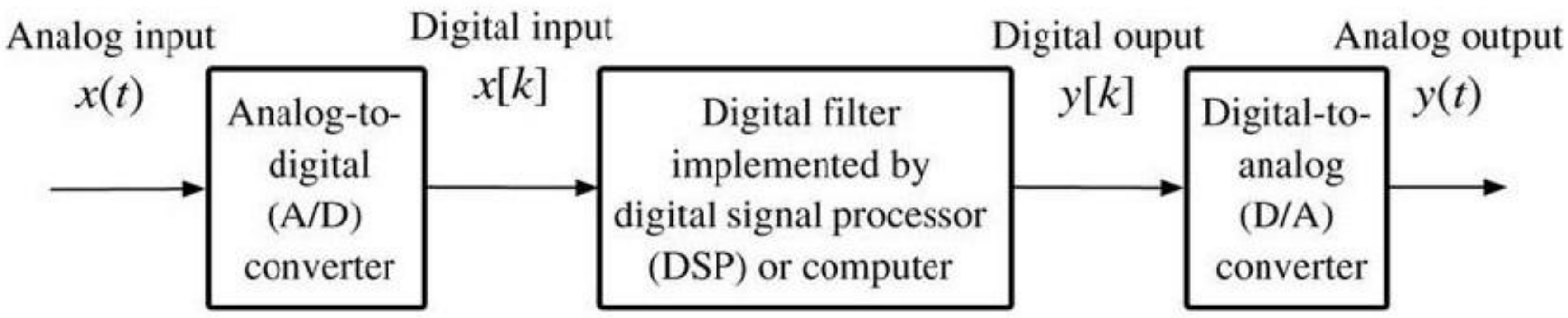
- *The bandwidth over which the amplitude response remains constant within 3dB.*
- *cut-off frequency*





Digital Filters

Sampling, quantizing, and coding



Basic diagram of a digital filter in practical applications.

Linear Distortion

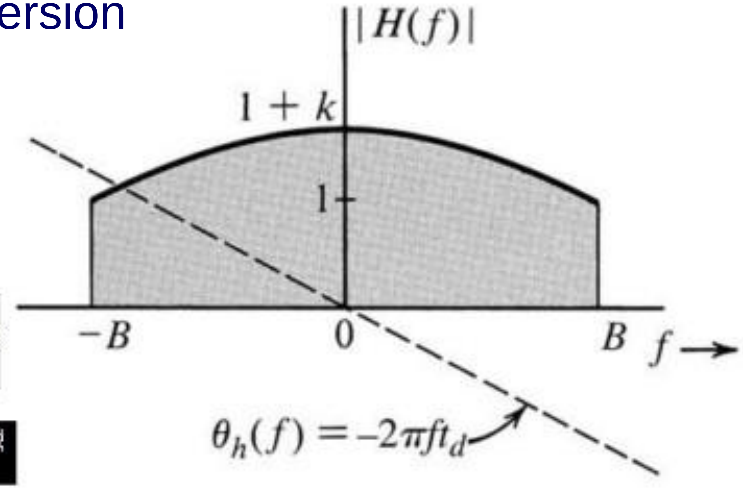
Magnitude distortion
Phase Distortion: Spreading/dispersion

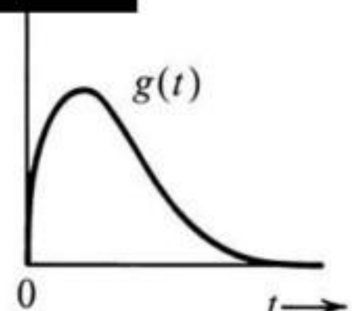
$$H(f) = \Pi\left(\frac{f}{2B}\right)(1 + k\cos 2\pi fT)e^{-j2\pi f t_d}$$

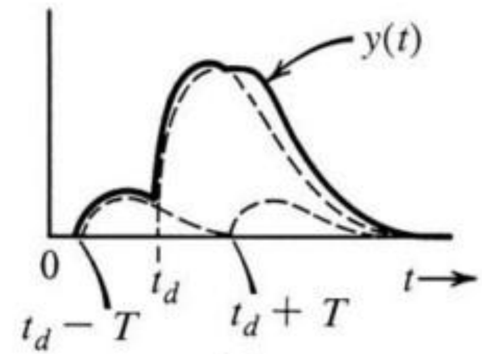
$$Y(f) = G(f)H(f) = G(f)\Pi\left(\frac{f}{2B}\right)(1 + k\cos 2\pi fT)e^{-j2\pi f t_d}$$

$$Y(f) = G(f)e^{-j2\pi f t_d} + k\left[G(f)\cos 2\pi fT\right]e^{-j2\pi f t_d}$$

(a)







$$k\left[G(f)\cos 2\pi fT\right]e^{-j2\pi f t_d} = k\,\frac{e^{j2\pi fT} + e^{-j2\pi fT}}{2}\,G(f)\,e^{-j2\pi f t_d} = \frac{k}{2}\left(e^{-j2\pi f(t_d - T)} + e^{-j2\pi f(t_d + T)}\right)$$

(b)

$$y(t) = g(t - t_d) + \frac{k}{2}\left[g(t - t_d - T) + g(t - t_d + T)\right]$$

(c)

Pulse is dispersed when it passes through a system that is not distortionless.

Distortion Caused by Channel Nonlinearities

$$y(t) = a_0 + a_1 x(t) + a_2 x^2(t) + \cdots + a_k x^k(t)$$

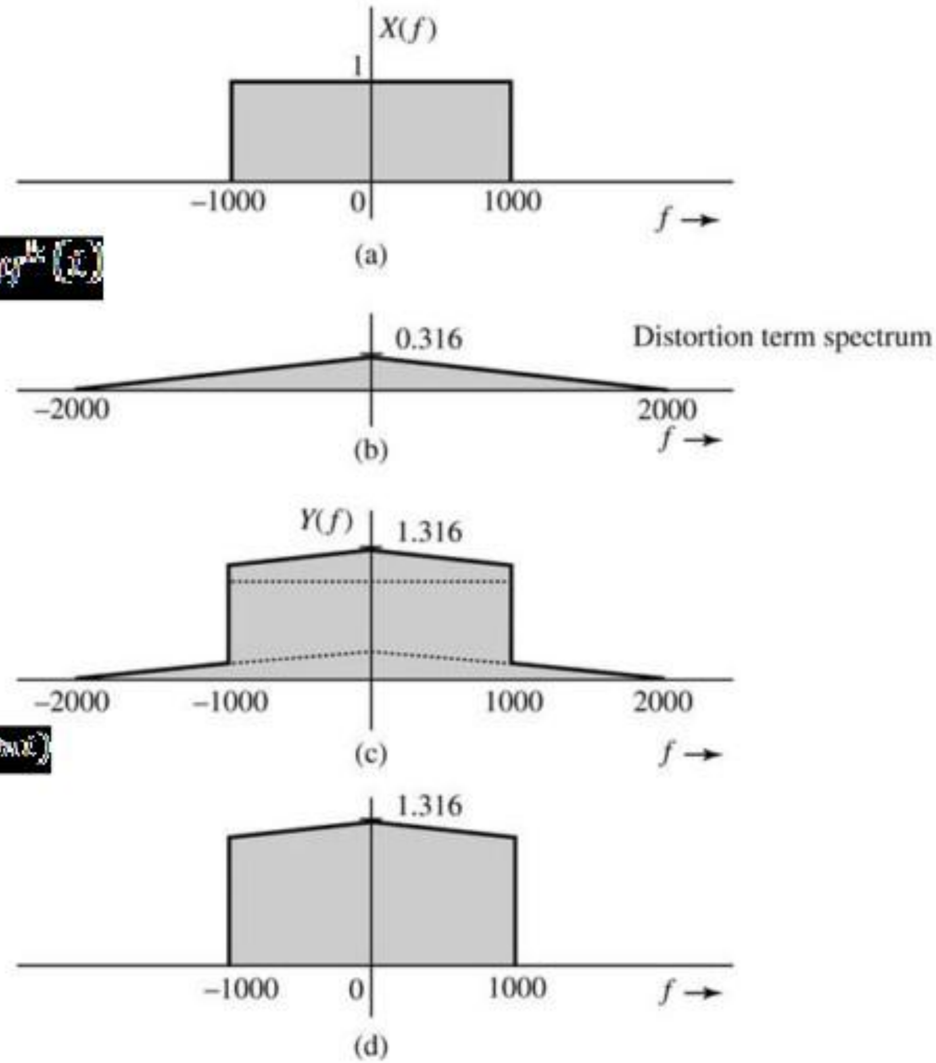
$$y(t) = x(t) + 0.000158\, x^2(t)$$

$$X(f) = \Pi\left(\frac{f}{2000}\right)$$

$$x(t) = 2000\,\text{sinc}(2000\pi t)$$

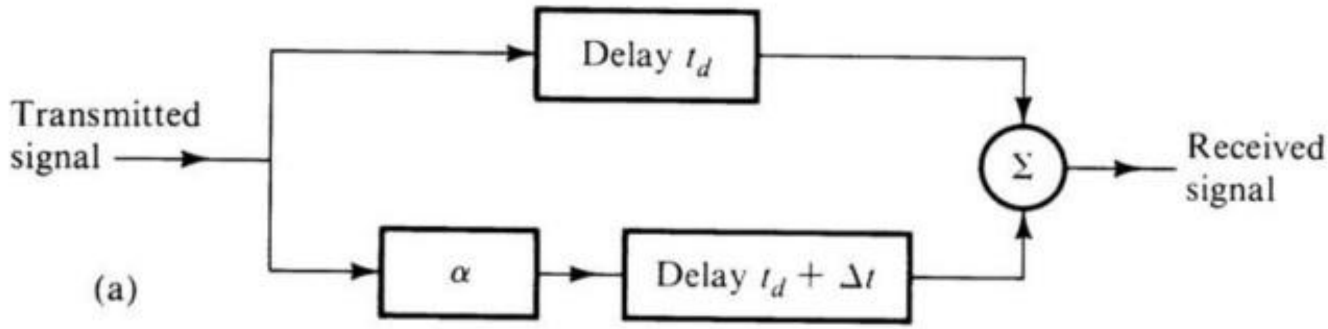
$$y(t) = 2000\,\text{sinc}(2000\pi t) + 0.316 \times 2000\,\text{sinc}^2(2000\pi t)$$

$$Y(f) = \Pi\left(\frac{f}{2000}\right) + 0.316\,\Lambda\left(\frac{f}{4000}\right)$$



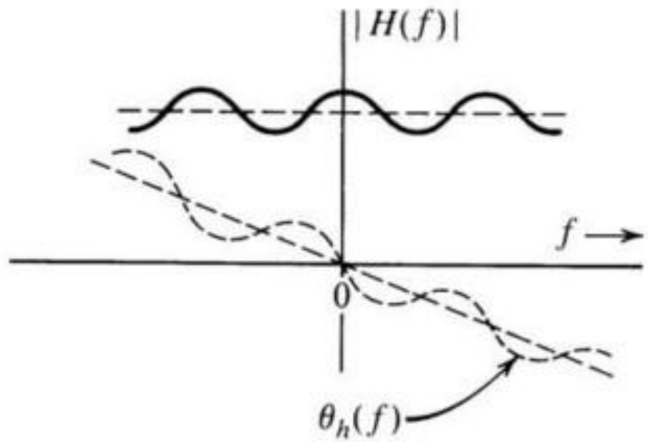
Signal distortion caused by nonlinear operation: (a) desired (input) signal spectrum; (b) spectrum of the unwanted signal (distortion) in the received signal; (c) spectrum of the received si (d) spectrum of the received signal after low-pass filtering.

Multipath Effects



(a)

$$H(f) = e^{-j2\pi f t_d} + \alpha e^{-j2\pi f (t_d + \Delta t)}$$



(b)

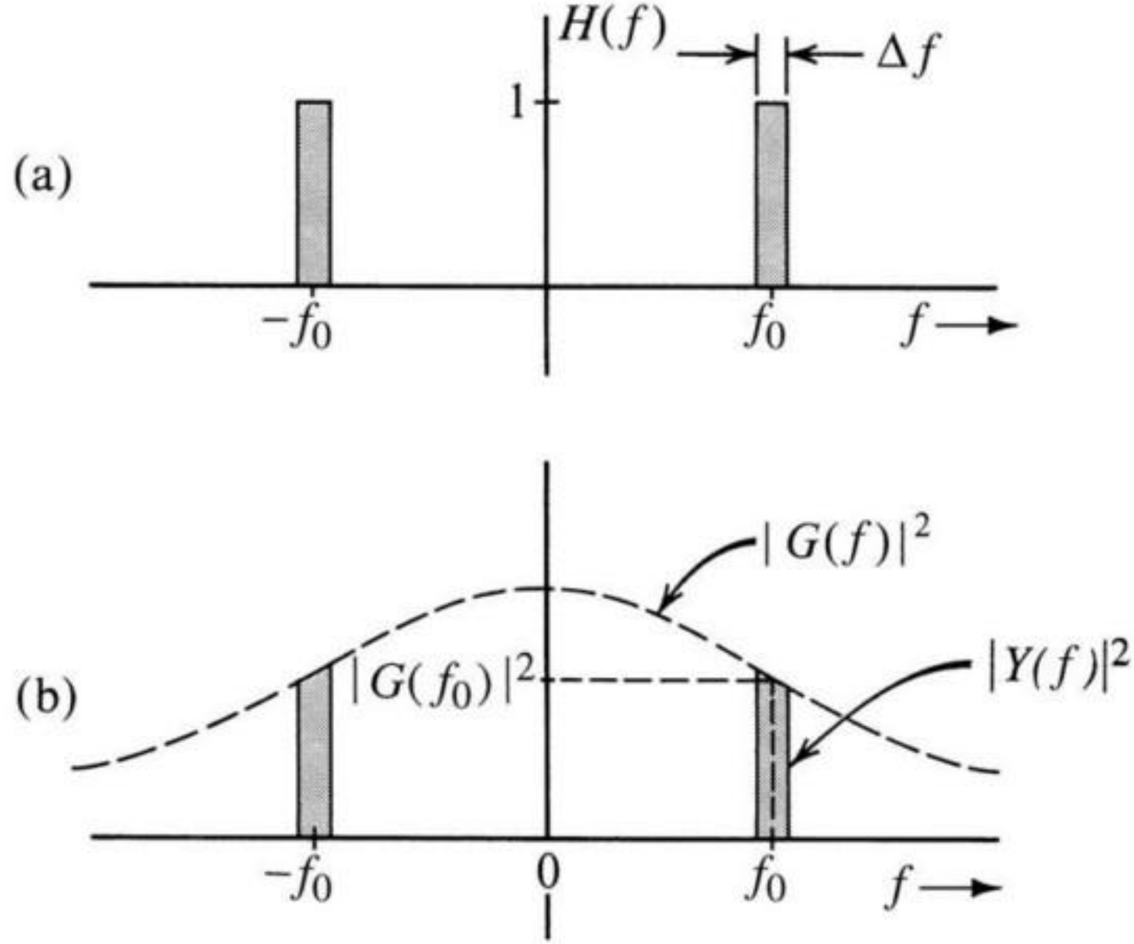
Multipath transmission.

Signal Energy: Parseval's Theorem

$$E_g = \int_{-\infty}^{+\infty} |g(t)|^2 dt = \int_{-\infty}^{+\infty} |G(f)|^2 df$$

Energy Spectral Density

$$\Psi_g(f) = |G(f)|^2$$



Interpretation of the energy spectral density of a signal.

Essential Bandwidth: the energy content of the components of frequeicies greater than B Hz is negligible.

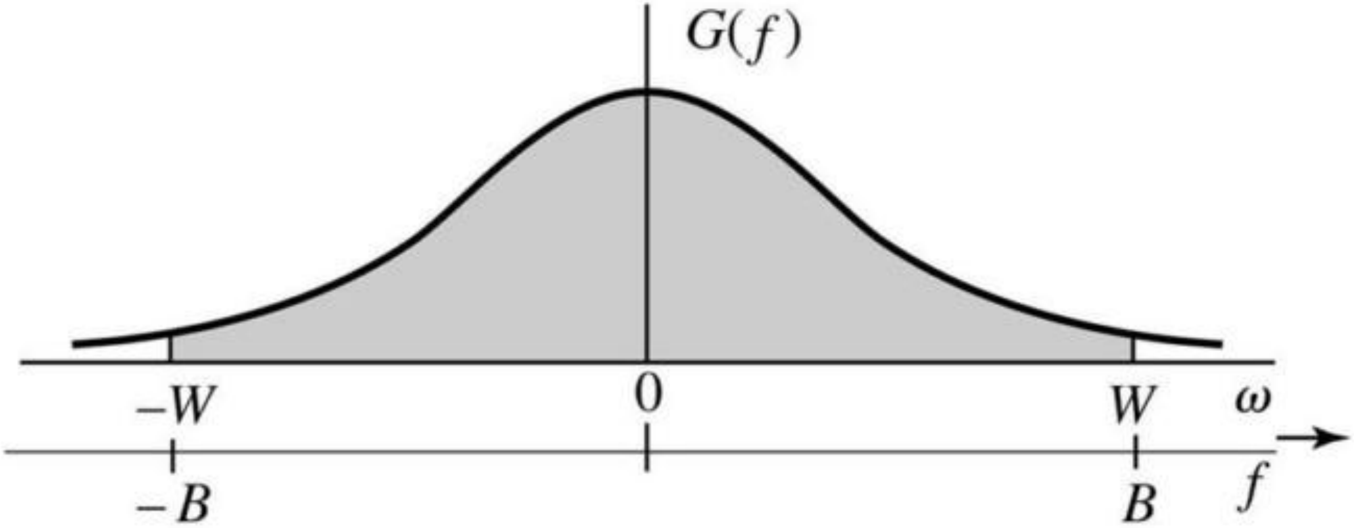
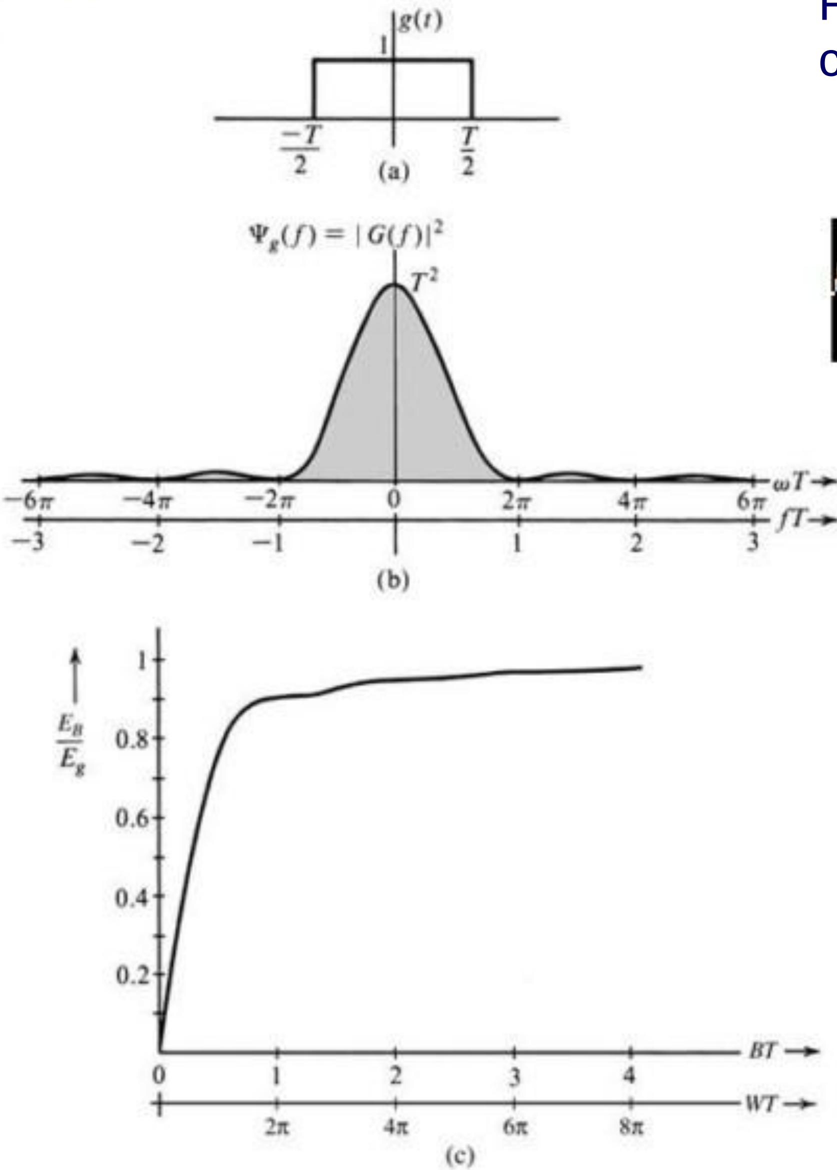


Figure Estimating the essential bandwidth of a signal.

Find the essential bandwidth where it contains at least 90% of the pulse energy.



$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-T/2}^{T/2} 1\, dt = T$$

$$\Pi\left(\frac{t}{T}\right) \Leftrightarrow T\,\text{sinc}(\pi f T)$$

$$\Psi_g(f) = T^2\,\text{sinc}^2(\pi f T)$$

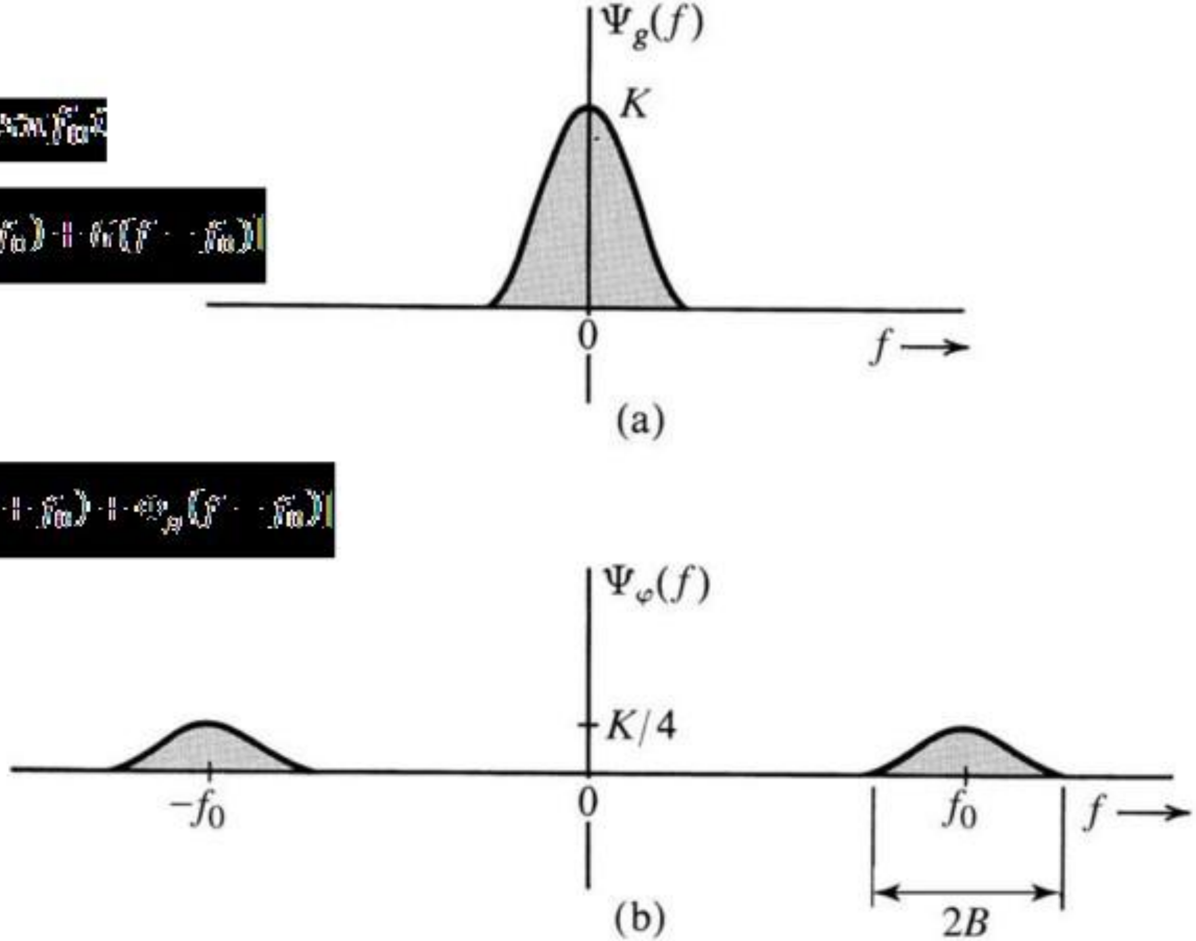
$$E_B = \int_{-B}^{B} T^2\,\text{sinc}^2(\pi f T)\, df$$

Energy of Modulated Signals

$$\varphi(t) = g(t)\cos 2\pi f_0 t$$

$$\Phi(f) = \frac{1}{2}[G(f + f_0) + G(f - f_0)]$$

$$\Psi_{\varphi}(f) = \frac{1}{4}[\Psi_g(f + f_0) + \Psi_g(f - f_0)]$$



Energy spectral densities of (a) modulating and (b) modulated signals.

Determine the ESD of $g(t) = e^{-at}u(t)$

Autocorrelation Function

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t+\tau)\,dt$$

$$\psi_g(\tau) \Longleftrightarrow \Psi_g(f)$$

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)\,dt = e^{a\tau}\int_{\tau}^{\infty} e^{-2at}\,dt = \frac{1}{2a}e^{-a\tau}$$

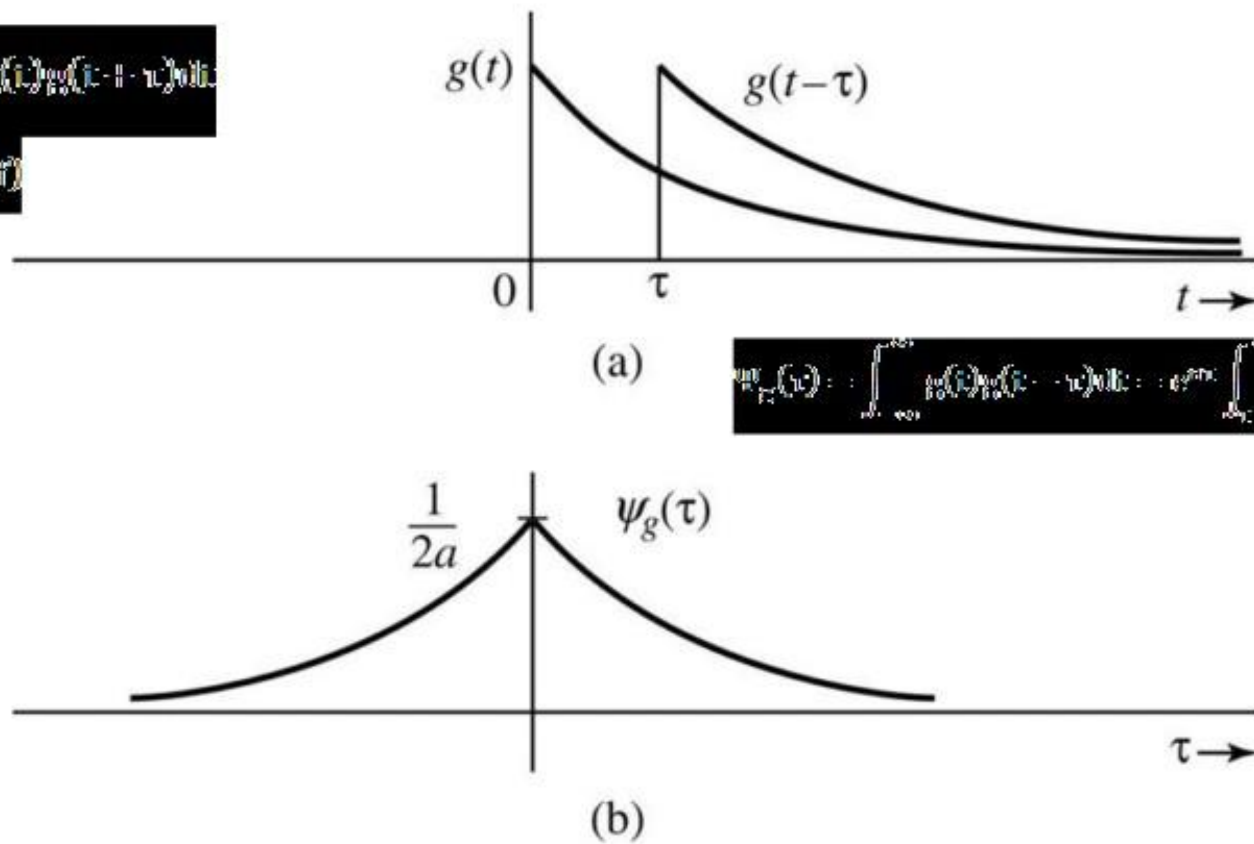
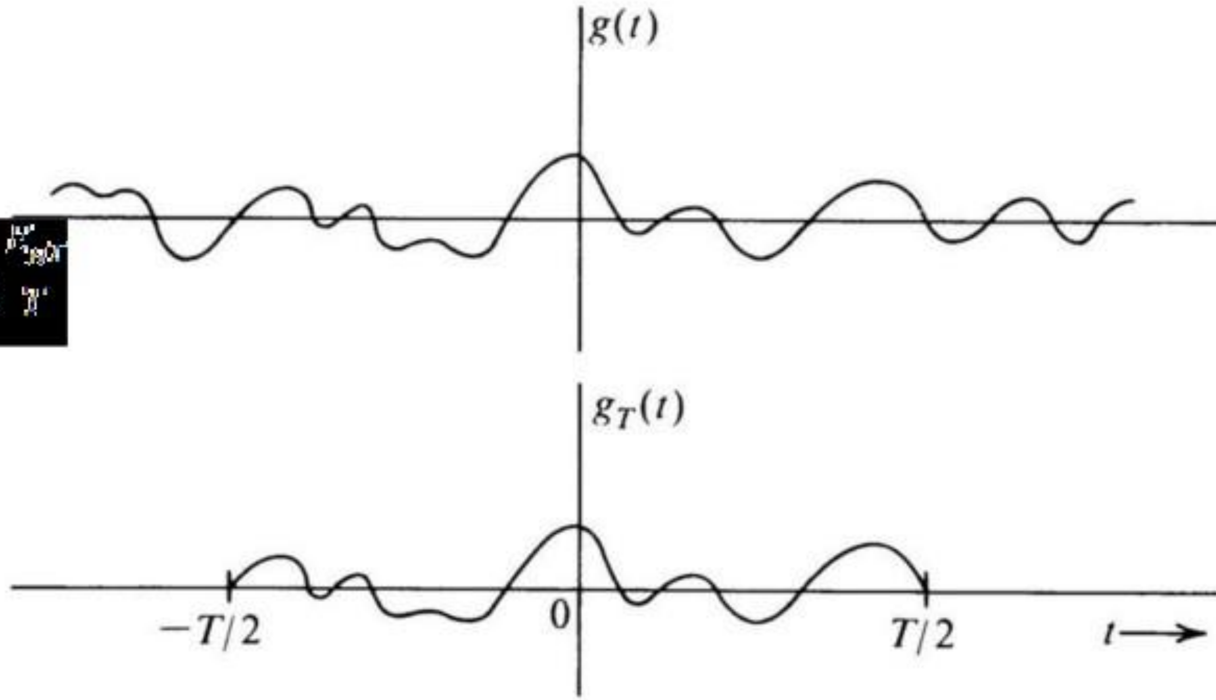


Figure Computation of the time autocorrelation function.

Signal Power

$$P_g = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g^2(t)\,dt = \lim_{T\to\infty} \frac{E_{g_T}}{T}$$



Limiting process in derivation of PSD.

Power Spectral Density

$$S_g(f) = \lim_{T\to\infty} \frac{|G_T(f)|^2}{T}$$

Time Autocorrelation Function of Power Signals

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(t)g(t-\tau)\,dt$$

PSD of Modulated Signals

$$S_s(f) = \frac{1}{4}[S_g(f+f_c) + S_g(f-f_c)]$$

Convolution Representation

DT Unit-Impulse Response

- Consider the DT SISO system:

- If the input signal is $x[n] = \delta[n]$ and the system has no energy at $n = 0$, the output $y[n] = h[n]$ is called the impulse response f the system

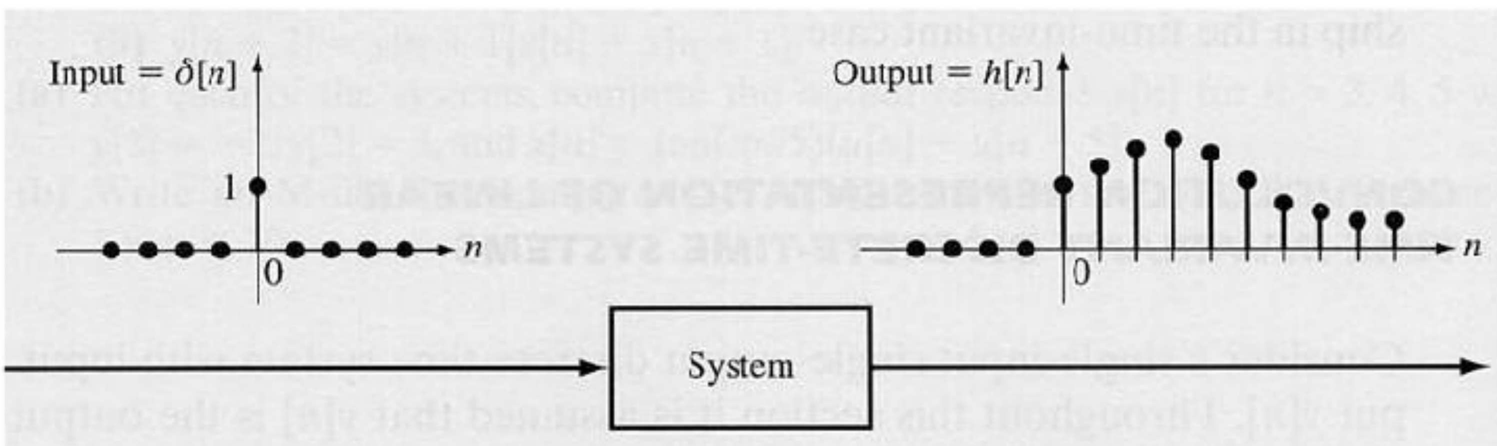
Example

- Consider the DT system described by

$$y[n] + ay[n-1] = bx[n]$$

- Its impulse response can be found to be

$$h[n] = \begin{cases} (-a)^n b, & n = 0, 1, 2, \ldots \\ 0, & n = -1, -2, -3, \ldots \end{cases}$$



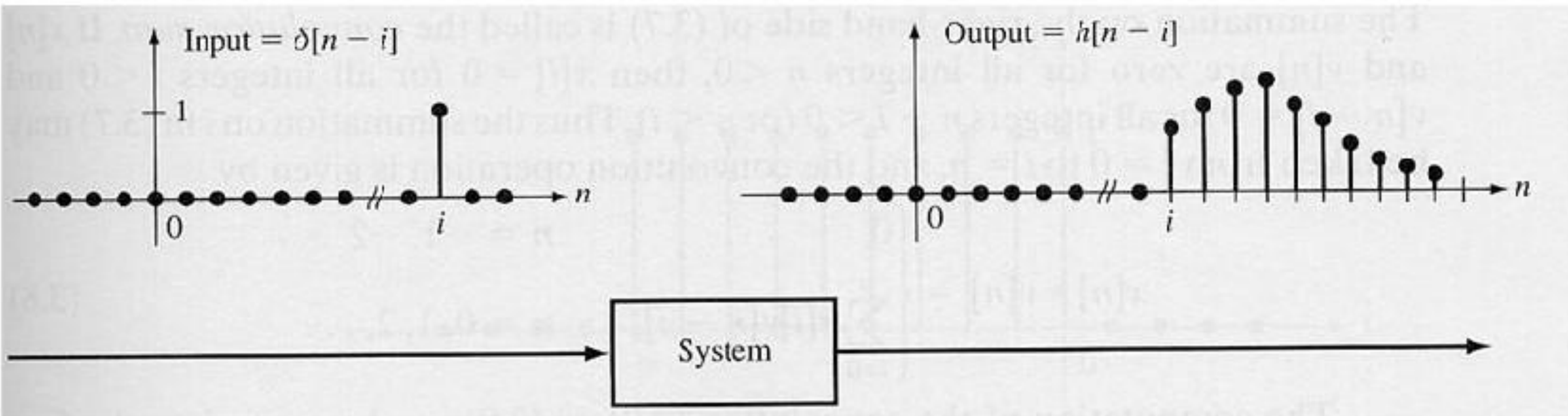
Representing Signals in Terms of Shifted and Scaled Impulses

- Let $x[n]$ be an arbitrary input signal to a DT LTI system
- Suppose that $x[n] = 0$ for $n = -1, -2, \ldots$
- This signal can be represented as

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

$$= \sum_{i=0}^{\infty} x[i]\delta[n-i], \qquad n = 0, 1, 2, \ldots$$

Exploiting Time-Invariance and Linearity



$$y[n] = \sum_{i=0}^{\infty} x[i]h[n-i], \qquad n \geq 0$$

The Convolution Sum

- This particular summation is called the **convolution sum**

$$y[n] = \sum_{i=-\infty}^{\infty} x[i]h[n-i]$$

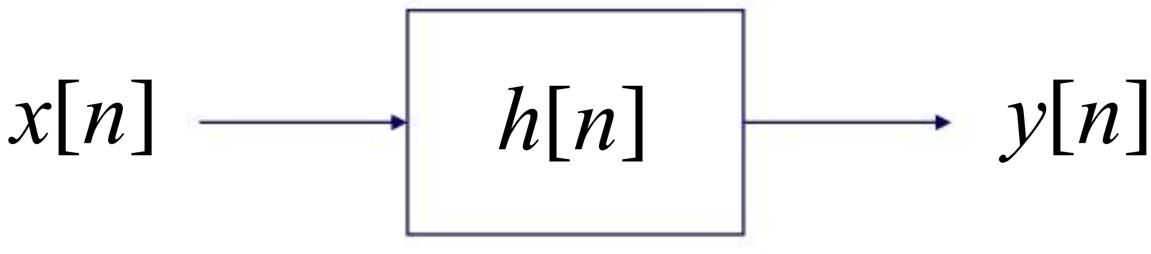
$$x[n] * h[n]$$

- Equation $y[n] = x[n] * h[n]$ is called the *convolution representation of the system*

- Remark: a DT LTI system is completely described by its impulse response *h*[*n*]

Block Diagram Representation of DT LTI Systems

- Since the impulse response $h[n]$ provides the complete description of a DT LTI system, we write



The Convolution Sum for Noncausal Signals

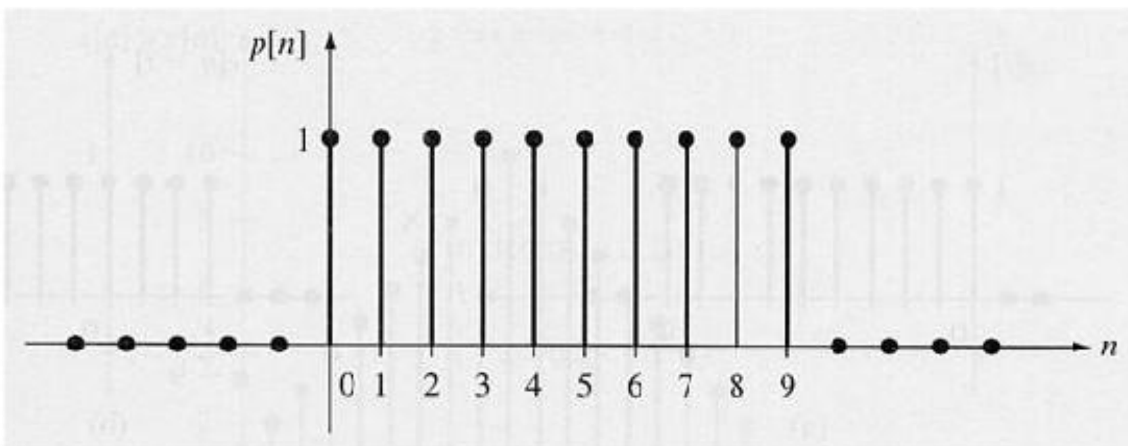
and *v*[*n*] that are not zero for negative times **noncausal signals**

- Then, their convolution is expressed by the two-sided series

$$y[n] = \sum_{i=-\infty}^{\infty} x[i]v[n-i]$$

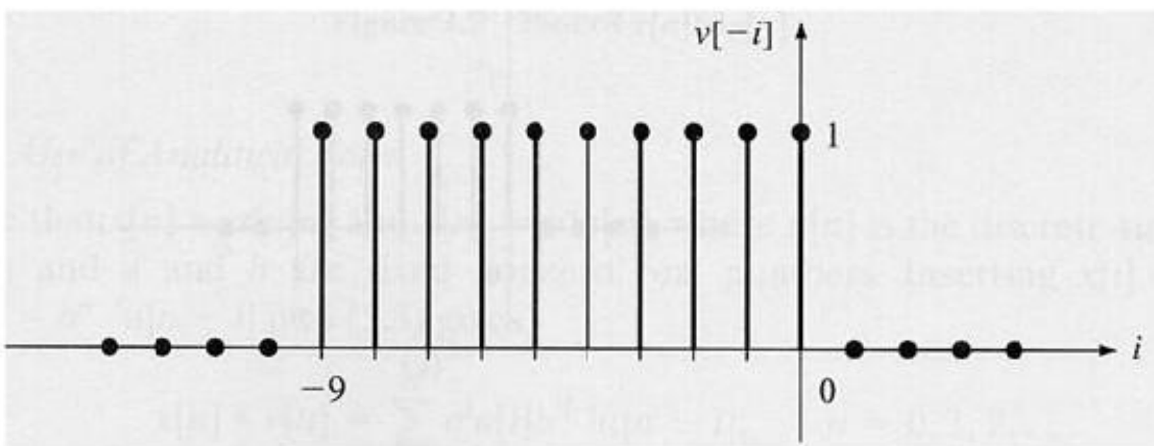
Example: Convolution of Two Rectangular Pulses

- Suppose that both $x[n]$ and $v[n]$ are equal to the rectangular pulse $p[n]$ (causal signal) depicted below

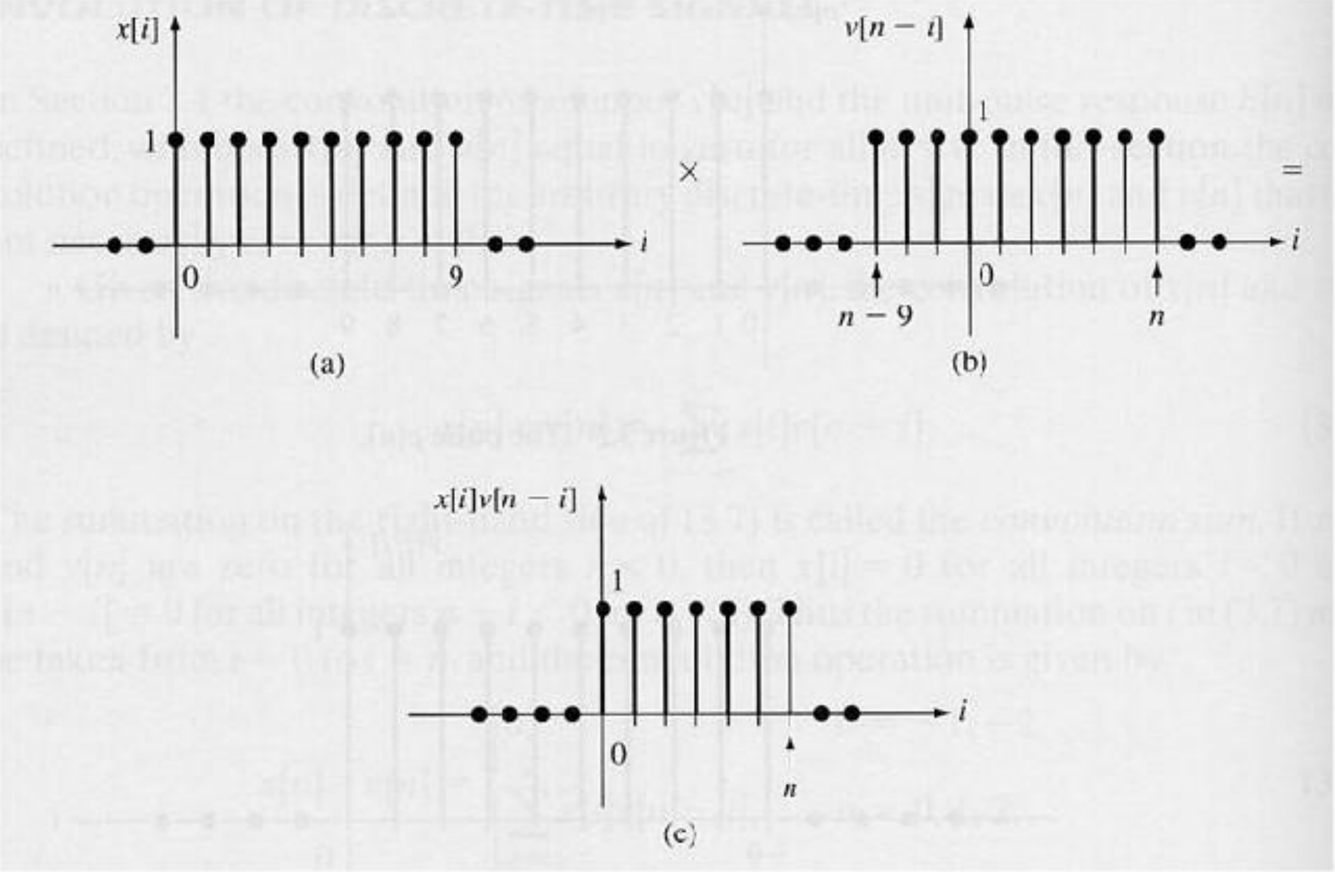


The Folded Pulse

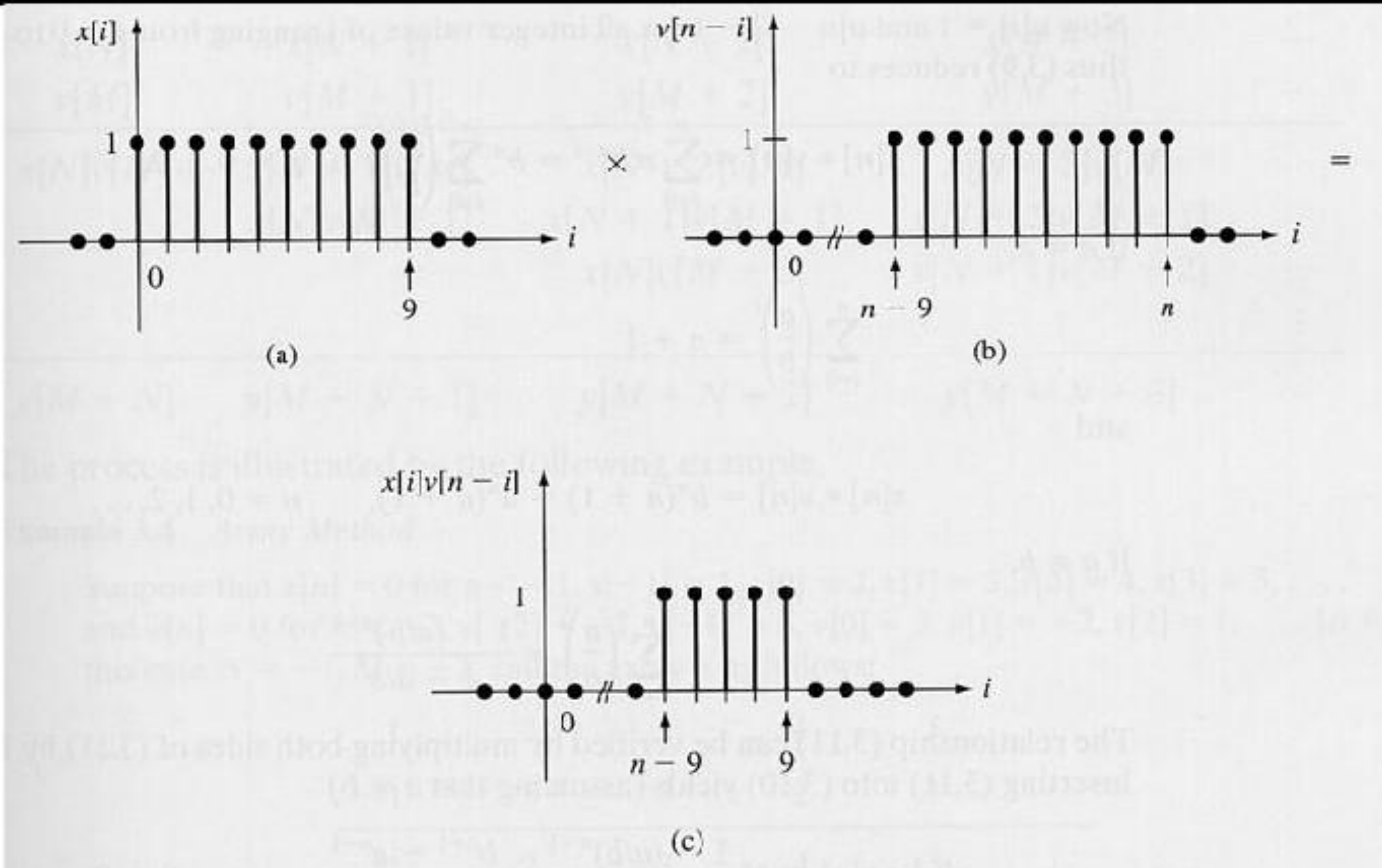
- The signal $v[-i]$ is equal to the pulse $p[i]$ folded about the vertical axis



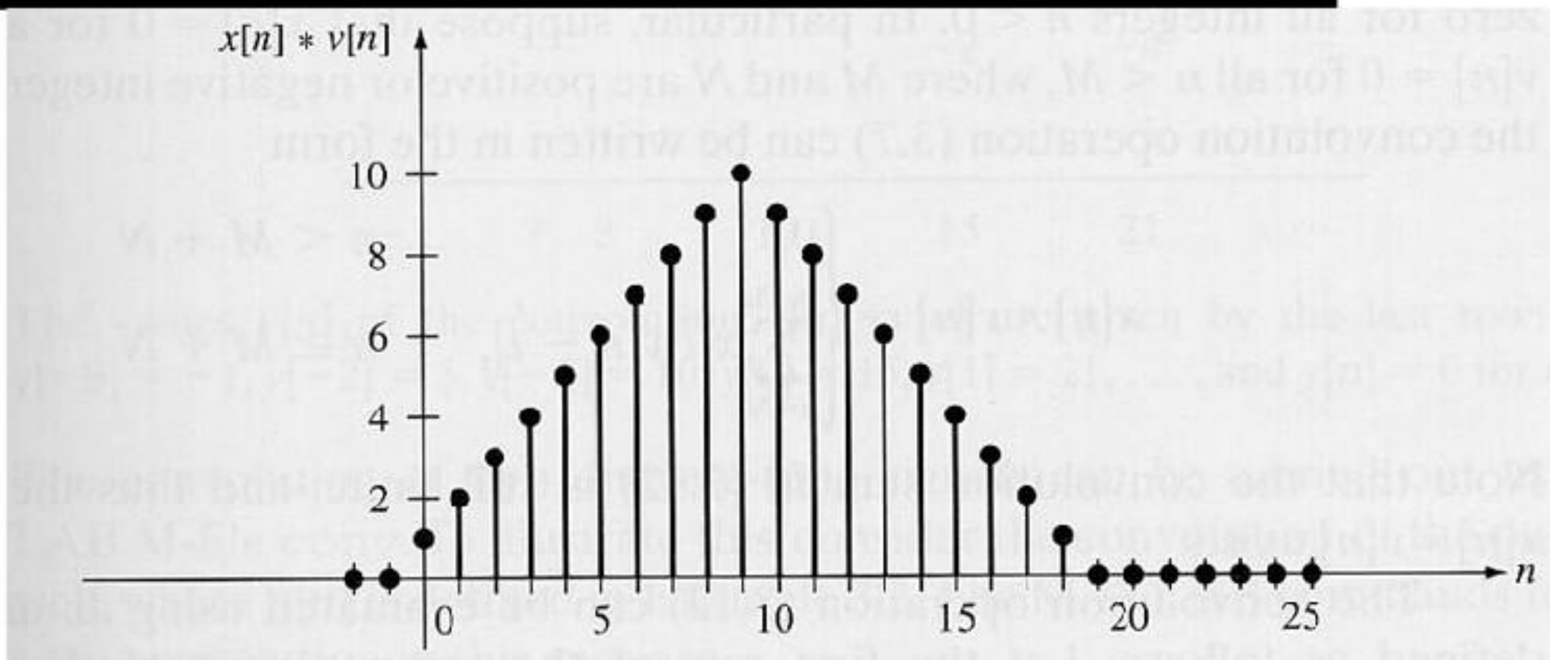
Sliding $v[n-i]$ over $x[i]$



Sliding $v[n-i]$ over $x[i]$ - Cont'd



Plot of $x[n] * v[n]$



Properties of the Convolution Sum

- Associativity

$$x[n]\cdot \quad (v[n]\cdot \quad w[n]) \cdot \quad (x[\; v \quad n]) \cdot \quad w[n]$$

- Commutativity

$$x[n]\cdot \quad v[n] \cdot \quad v[n]\cdot \quad x[n]$$

- Distributivity w.r.t. addition

$$x[n]\cdot \quad (v[n] \cdot \quad w[n]) \cdot x[n \quad v \quad n]\cdot \quad x[n]\cdot \quad w[n]$$

Properties of the Convolution Sum - Cont'd

- Shift property: efine

$$x_q[n] = x[n-q]$$

$$v_q[n] = v[n-q]$$

$$w[n] = x[n] * v[n]$$

then

$$w[n-q] = x_q[n] * v[n] = x[n] * v_q[n]$$

- Convolution with the unit impulse

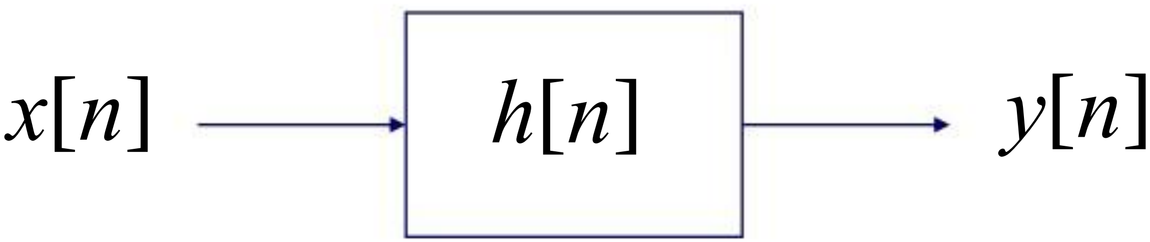
$$x[n] * \delta[n] = x[n]$$

- Convolution with the shifted unit impulse

$$x[n] * \delta_q[n] = x[n-q]$$

Example: Computing Convolution with *Matlab*

- Consider the DT LTI system



- impulse response:

$$h[n] = \sin(0.5n), \quad n \geq 0$$

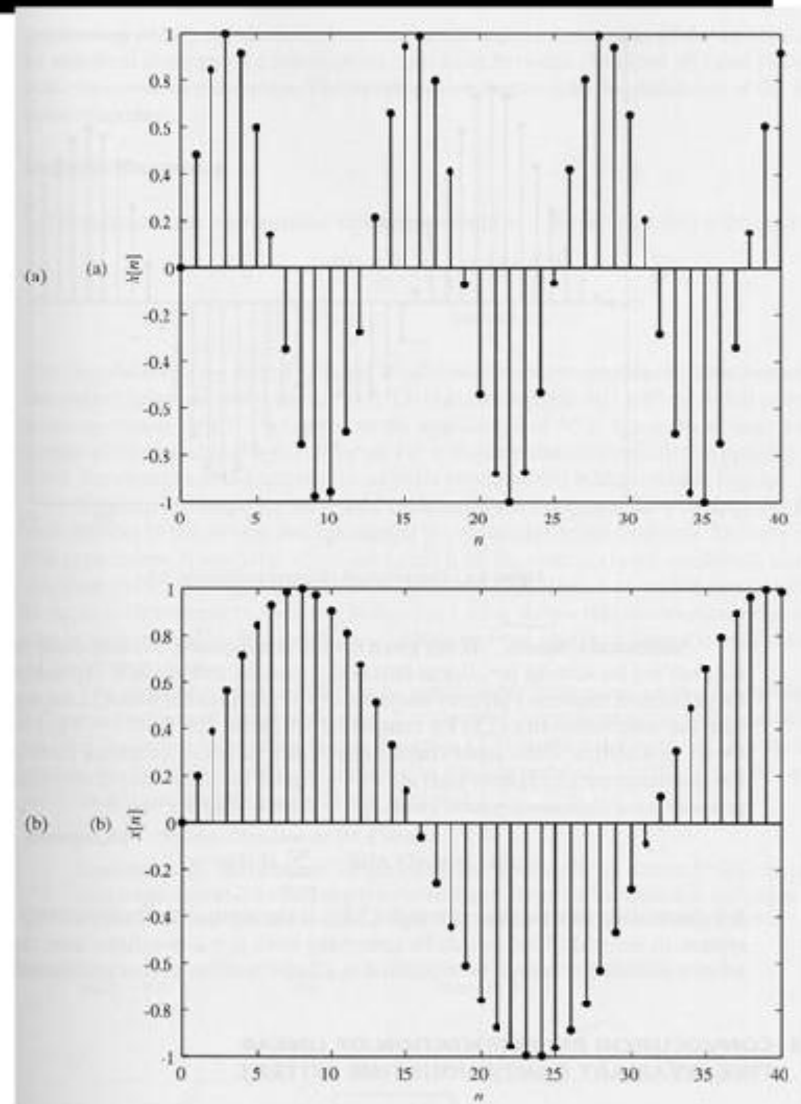
- input signal:

$$x[n] = \sin(0.2n), \quad n \geq 0$$

Example: Computing Convolution with *Matlab* – Cont'd

$h[n] \cdot \sin(0.5n), \quad n \cdot 0$

$x[n] \cdot \sin(0.2n), \quad n \cdot 0$



Example: Computing Convolution with *Matlab* – Cont'd

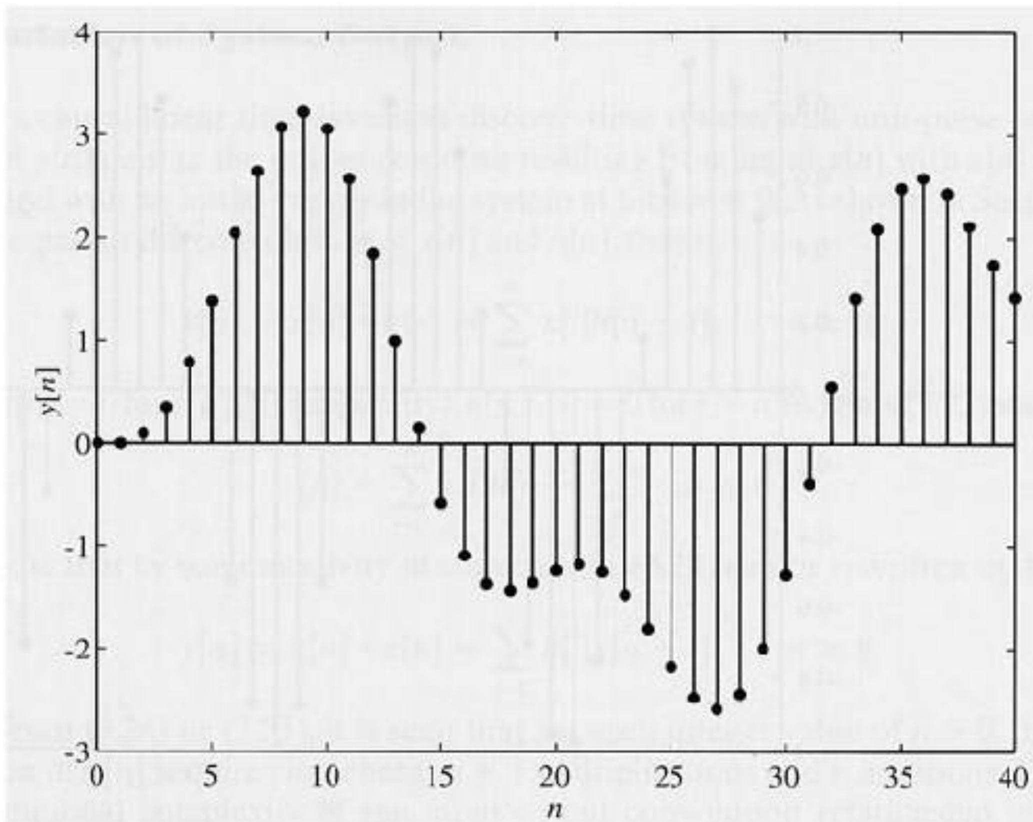
- *Matlab* code:

$$n \cdot \quad 0,1,\cdot \quad ,40$$

```
n=0:40;
x=sin(0.2*n);
h=sin(0.5*n);
y=conv(x,h);
stem(n,y(1:length(n)))
```

Example: Computing Convolution with *Matlab* – Cont'd

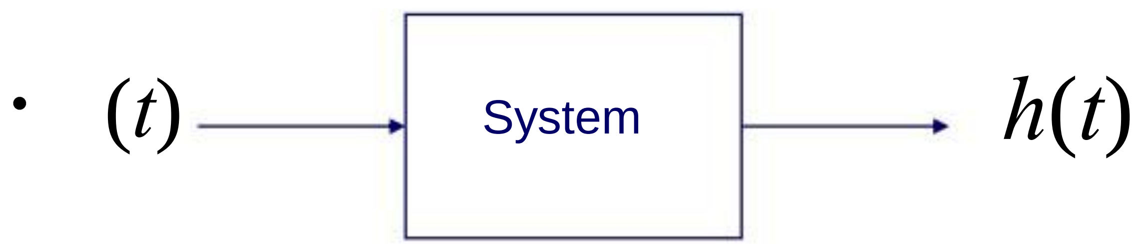
$$y[n] \cdot \quad x[n] \cdot \quad h[n]$$



CT Unit-Impulse Response

- Consider the CT SISO system:

- If the input signal is $x(t) \cdot \quad \cdot \quad (t)$ and the system has no energy at $t \cdot \; 0$ $\cdot$, the output $y(t) \cdot \; h(t)$ is called the impulse response of the system



Exploiting Time-Invariance

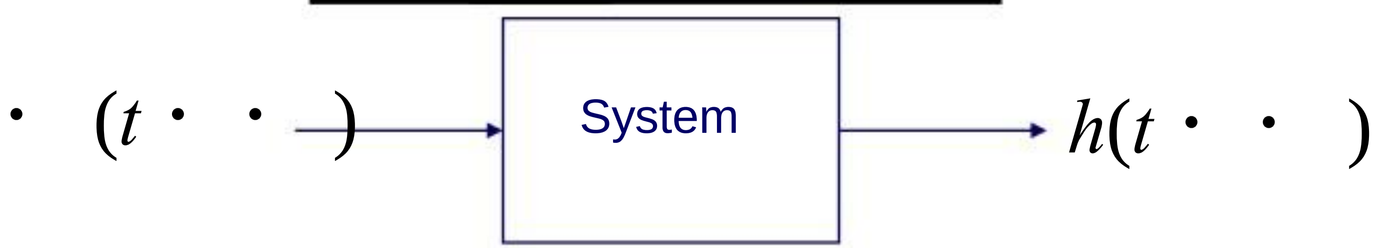
- Let *x*[*n*] be an arbitrary input signal with

$$x(t) \cdot \quad 0, \text{for } t \cdot \quad 0$$

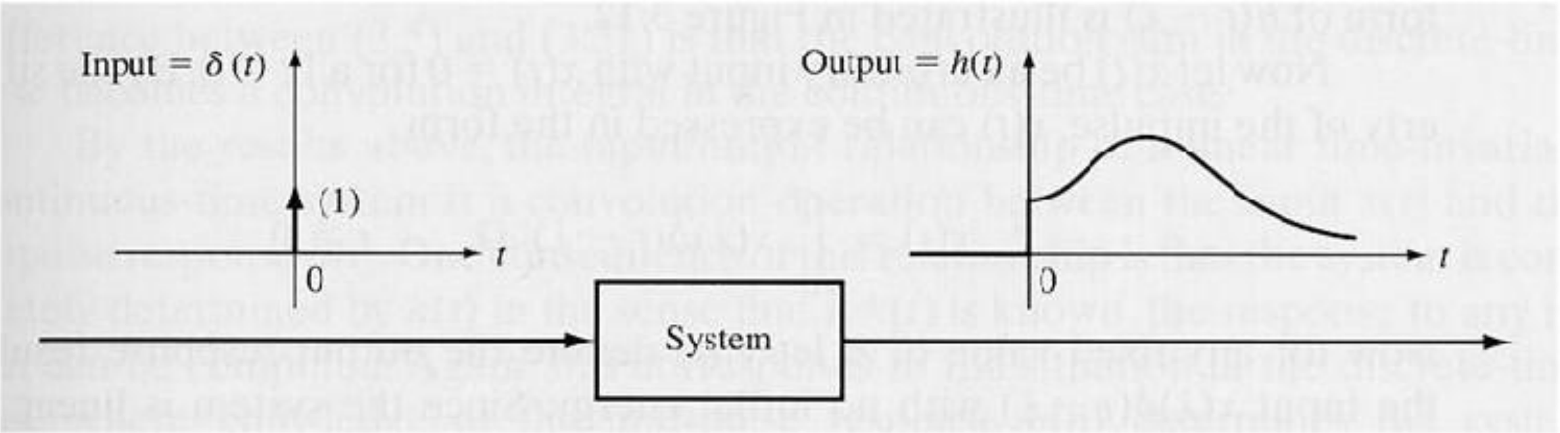
- Using the sifting property $\cdot \quad (t)$, we may write

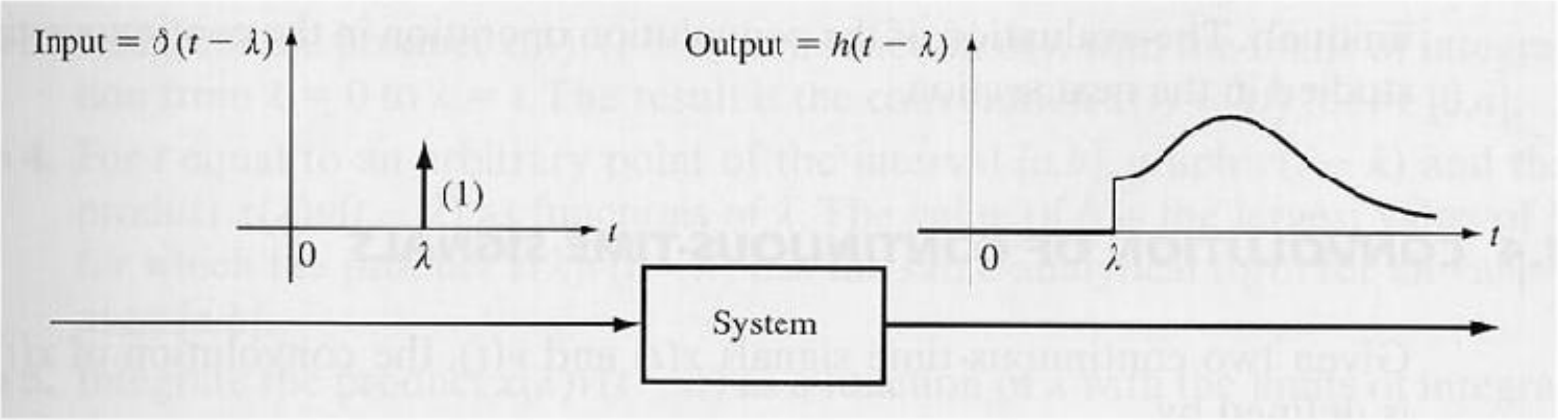
$$x(t) \cdot \int_{0\cdot}^{\cdot} x(\cdot)\cdot(t \cdot \cdot)d\cdot, \quad t \cdot \quad 0$$

- Exploiting time-invariance, t is



Exploiting Time-Invariance





Exploiting Linearity

- Exploiting linearity [redacted] is

$$y(t) = \int_{0^-}^{\infty} x(\tau)h(t-\tau)\,d\tau, \quad t \geq 0$$

- If the integrand $x(\tau)h(t-\tau)$ does not contain an impulse located at $\tau = 0$, the lower limit of the integral can be taken to be 0,i.e.,

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)\,d\tau, \quad t \geq 0$$

The Convolution Integral

- This particular integration is called the **convolution integral**

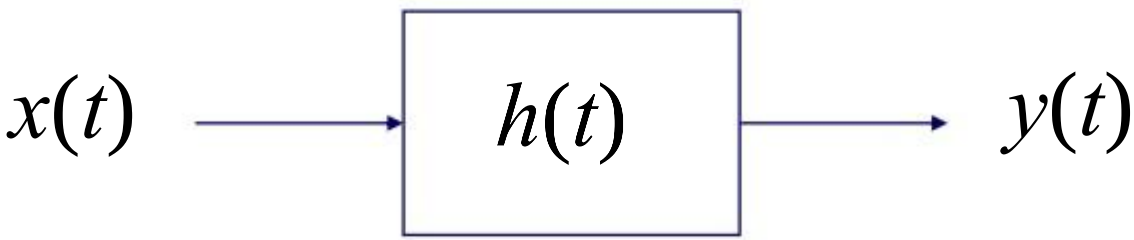
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau, \qquad t \ge 0$$

$$x(t) * h(t)$$

- Equation $y(t) = x(t) * h(t)$ is called the *convolution representation of the system*
- Remark: a CT LTI system is completely described by its impulse response $h(t)$

Block Diagram Representation of CT LTI Systems

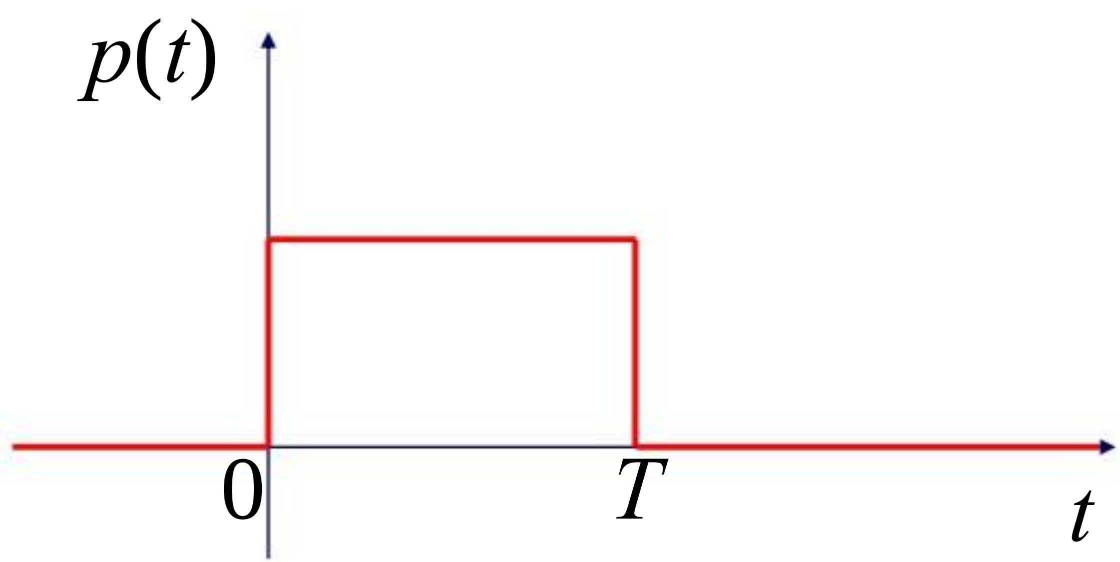
the complete description of a CT LTI system, we write



Example: Analytical Computation of the Convolution Integral

Suppose that $\qquad$ where

p(*t*) is the rectangular pulse depicted in figure $\quad x(t) \cdot h(t) \cdot p(t),$



Example – Cont'd

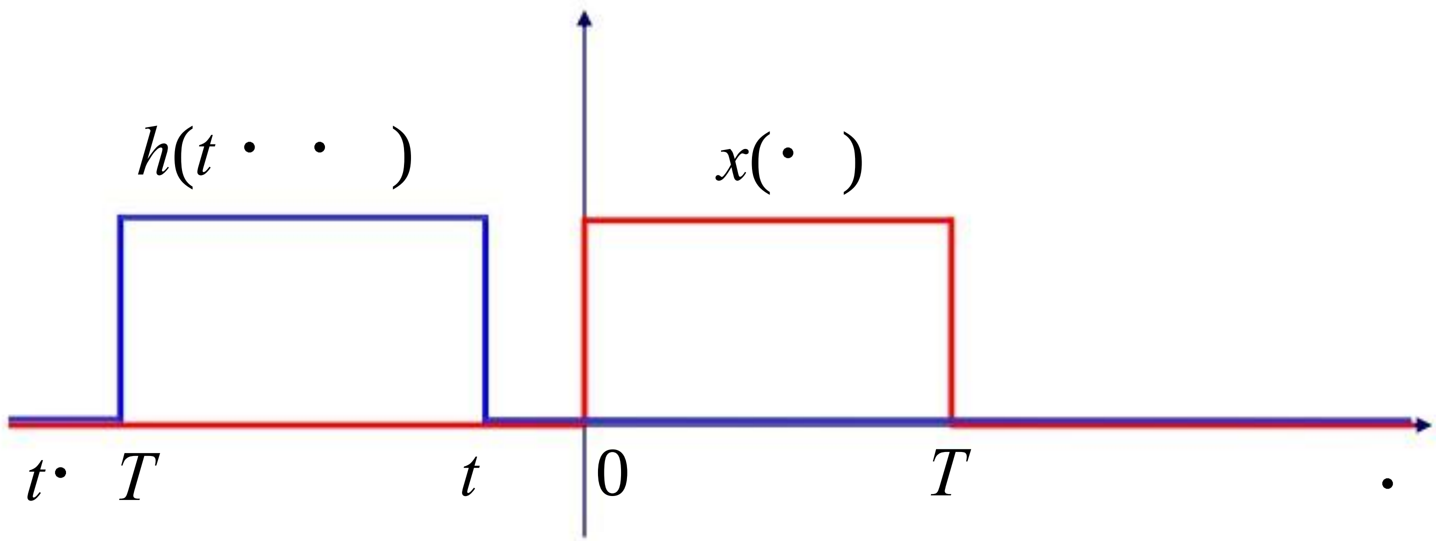
- In order to compute the convolution integral

$$y(t)=\int_0^\infty x(\tau)h(t-\tau)\,d\tau, \qquad t\ge 0$$

we have to consider four cases:

Example – Cont'd

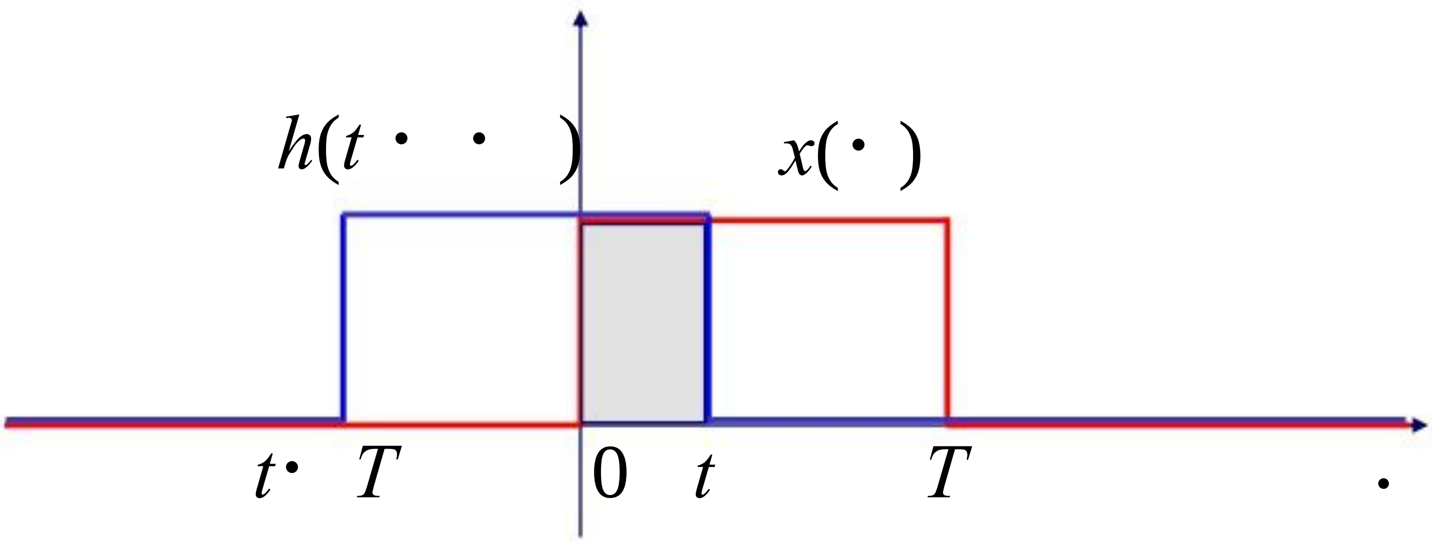
- Case 1: $t \cdot$ 0



$$y(t) \cdot \quad 0$$

Example – Cont'd

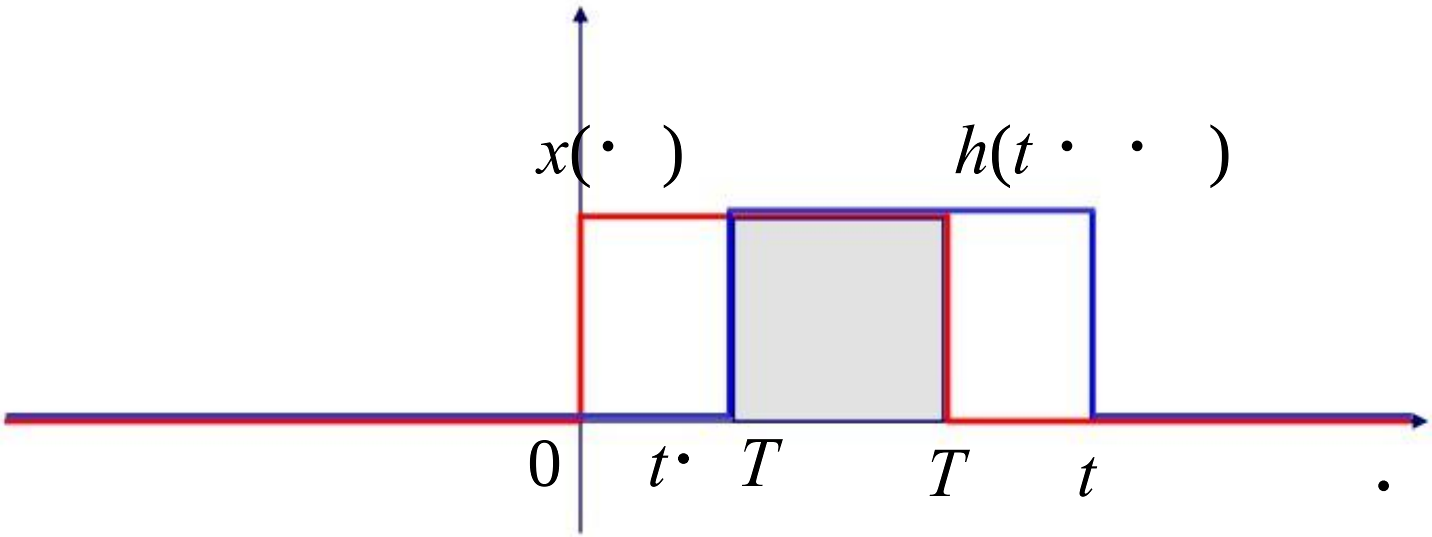
- Case 2: $0 \cdot \quad t \cdot \quad T$



$$y\;t)\cdot \bullet_{0}^{t}\; d\cdot \quad \cdot \quad t$$

Example – Cont'd

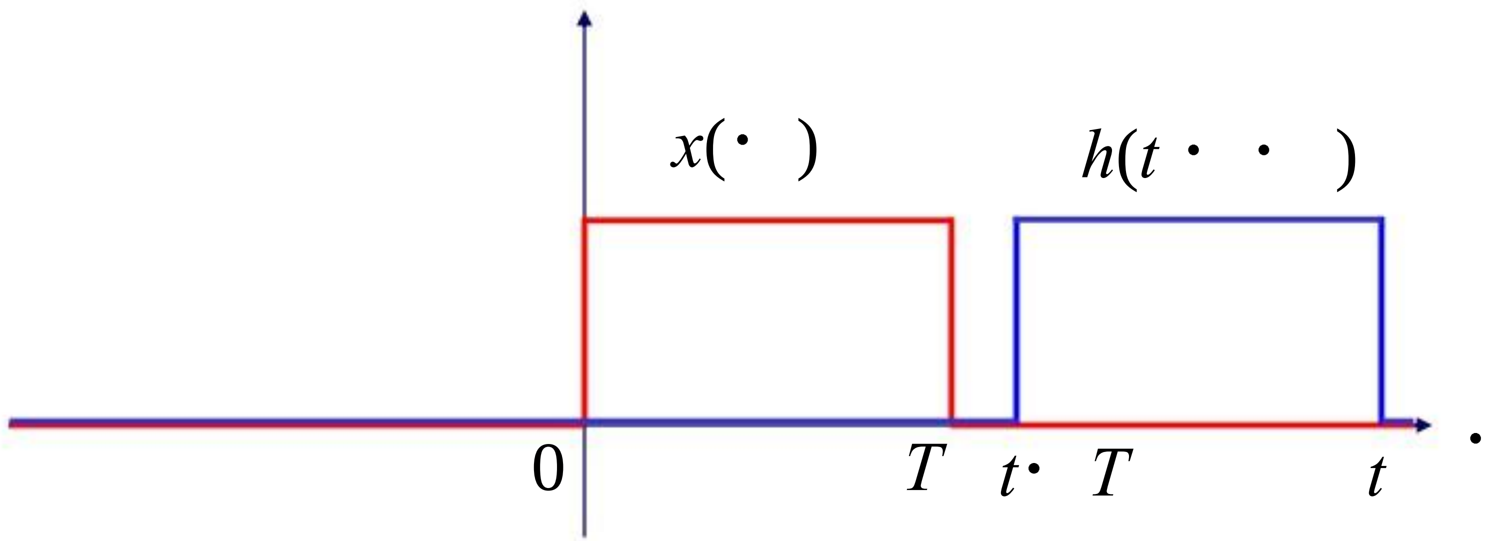
- Case 3: $0 \cdot \; t \cdot \; T \cdot \; T \quad \cdot \quad T \cdot \; t \cdot \; 2T$



$$y\ t) \cdot \int_{t \cdot \; T}^{T} d \cdot \; \cdot \; T \cdot \; (t \cdot \; T) \cdot \; 2T \cdot \; t$$

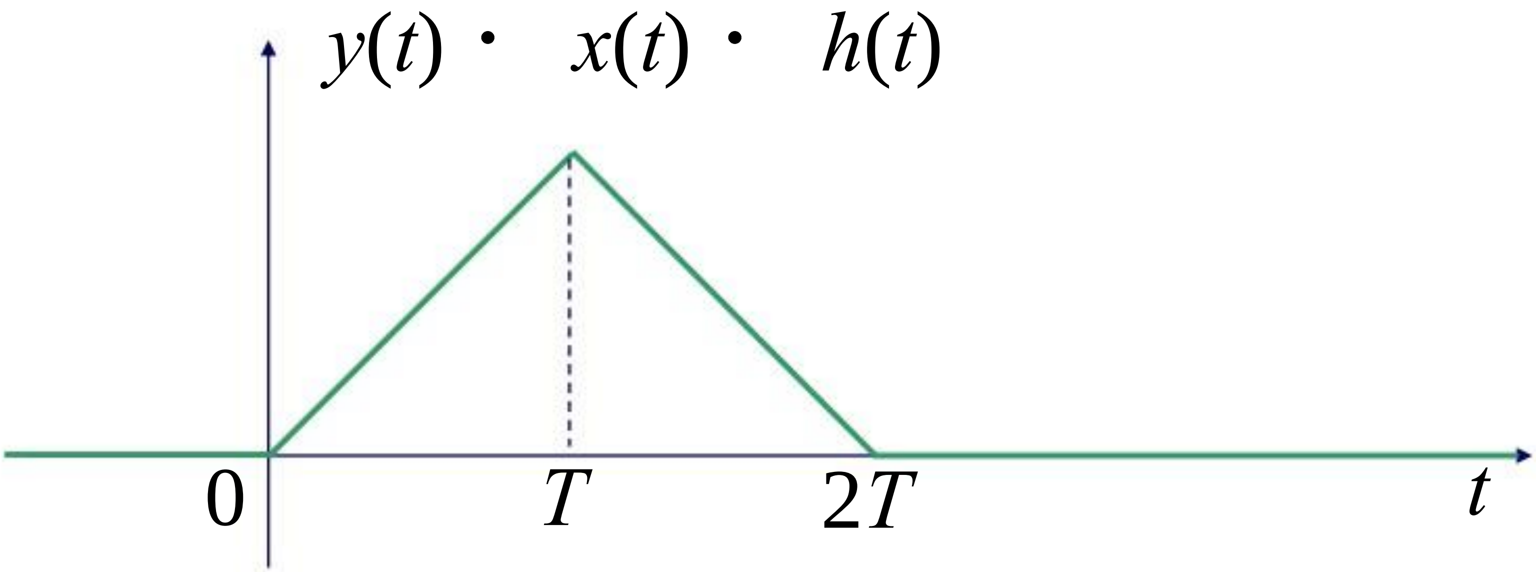
Example – Cont'd

- Case 4: $T < t - T \quad \Rightarrow \quad 2T < t$



$$y(t) = 0$$

Example – Cont'd



Properties of the Convolution Integral

- Associativity

$$x(t) \cdot (v(t) \cdot w(t)) \cdot (x(t) \cdot v(t)) \cdot w(t)$$

- Commutativity

$$x(t) \cdot v(t) \cdot v(t) \cdot x(t)$$

- Distributivity w.r.t. addition

$$x(t) \cdot (v(t) \cdot w(t)) \cdot x(t) \cdot v(t) \cdot x(t) \cdot w(t$$

Properties of the Convolution Integral - Cont'd

- **Shift property:** define

$$x_q(t) = x(t - q)$$

$$v_q(t) = v(t - q)$$

then

$$w(t) = x(t) * v(t)$$

$$w(t - q) = x_q(t) * v(t) = x(t) * v_q(t)$$

- **Convolution with the unit impulse**

$$x(t) * \delta(t) = x(t)$$

- **Convolution with the shifted unit impulse**

$$x(t) * \delta_q(t) = x(t - q)$$

Properties of the Convolution Integral - Cont'd

- **Derivative property:** the signal $x(t)$ is differentiable, then it is

$$\frac{d}{dt}\left[x(t)*v(t)\right]=\frac{dx(t)}{dt}*v(t)$$

- If both $x(t)$ and $v(t)$ are differentiable, then it is also

$$\frac{d^2}{dt^2}\left[x(t)*v(t)\right]=\frac{dx(t)}{dt}*\frac{dv(t)}{dt}$$

Properties of the Convolution Integral - Cont'd

- Integration property: define

$$x^{(-1)}(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

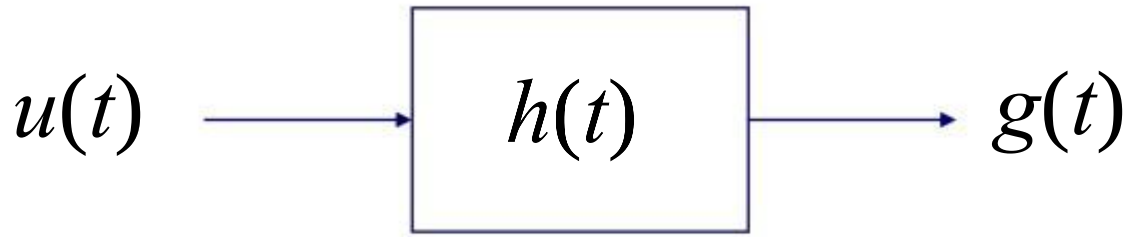
$$v^{(-1)}(t) = \int_{-\infty}^{t} v(\tau)\,d\tau$$

then

$$(x * v)^{(-1)}(t) = x^{(-1)}(t) * v(t) = x(t) * v^{(-1)}(t)$$

Representation of a CT LTI System in Terms of the Unit-Step Response

- Let $g(t)$ be the response of a system with impulse response $h(t)$ when $x(t) \cdot \; u(t)$ with no initial energy at time $t \cdot \; 0$, i.e.,



- Therefore, it is

$$g(t) \cdot \; h(t) \cdot \; u(t)$$

Representation of a CT LTI System in Terms of the Unit-Step Response – Cont'd

- Differentiating both sides

$$\frac{dg\ t)}{dt} \cdot \frac{dh\ t)}{dt} \cdot u\ t) \cdot\ h\ t) \cdot \qquad \frac{du\ t)}{dt}$$

- Recalling that

$$\frac{du\ t)}{dt} \cdot\ \cdot\ (t) \quad \text{and} \quad h(t) \cdot\ h(t) \cdot\ \cdot\ (t)$$

it is

$$\frac{dg\ t)}{dt} \cdot\ h\ t) \quad \text{or} \quad g\ t) \cdot\ \bullet_{0}^{t} h(\cdot\ )d\cdot$$

Definitions of the components/Keywords:

Convolution of two signals:

Let *x(t)* and *h(t)* are two continuous signals to be convolved.

The convolution of two signals is denoted by

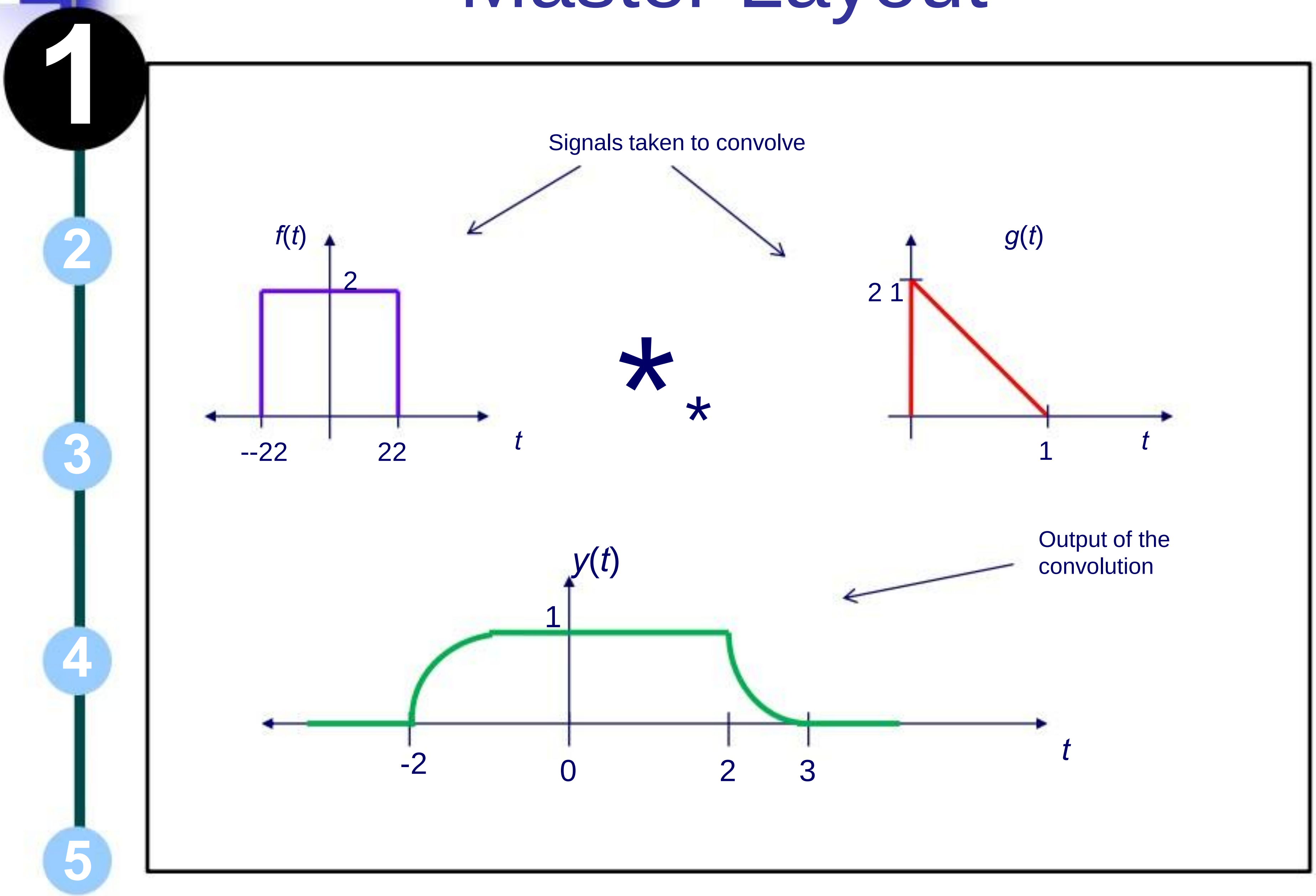
$$y(t) = x(t) * h(t)$$

which means

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

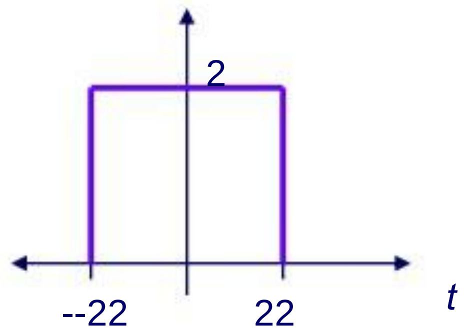
where • is the variable of integration.

Master Layout

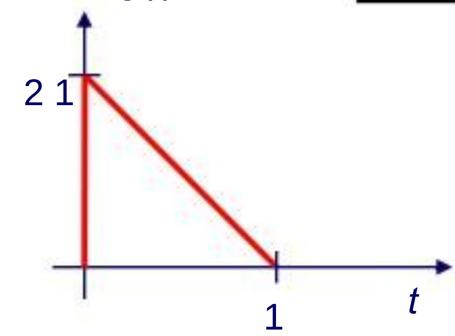


Step 1:

$f(t) = 2 \quad -2 \le t \le 2$



$g(t) = -t+1 \quad 0 \le t \le 1$



Instruction for the animator	Text to be displayed in the working area (DT)
• The first point in DT has to appear before the figures. • Then the blue figure has to appear. • After that the red figure has to appear. • After the figures, the next point in DT has to appear.	• *f(t)* and *g(t)* are the two continuous signals to be convolved. • The convolution of the signals is denoted by $y(t) = f(t) * g(t)$ which means $y(t) = \int_{-\infty}^{+\infty} f(x) g(t-x) dx$ where x is a dummy variable.

1 2 3 4 5

Step 2:

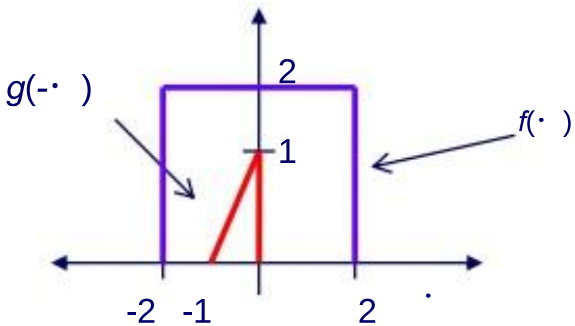


Fig. a

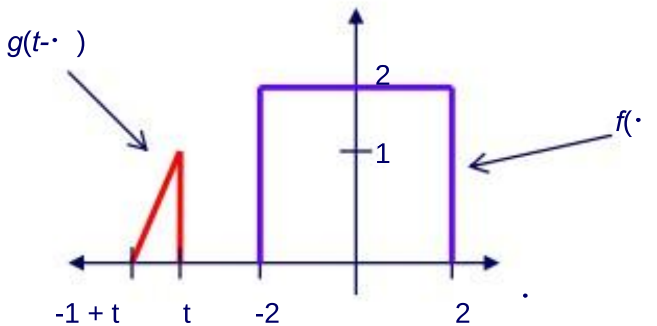
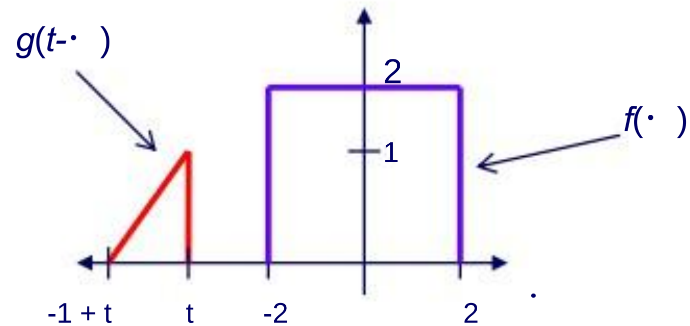


Fig. b

Instruction for the animator	Text to be displayed in the working area (DT)
• The figure in blue in fig. a has to appear then its label should appear. • Then the red figure has to appear. • After that the labeling of red figure has to appear. • In parallel to the fig. the text in DT has to appear. • First two sentences in DT has to appear with fig. a • The last sentence should appear with fig. b.	• The signal *f(· ·)* is shown • The reversed version of *g(·) i.e., g(-· · ·* is shown • The shifted version of g(-· · · *i.e., g(t-·)* is shown

Step 3: Calculation of *y(t)* in five stages

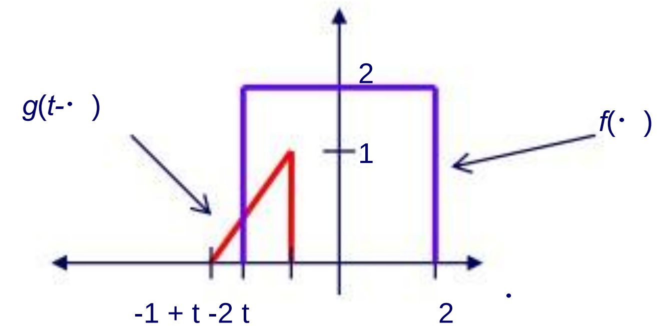
Stage - I : t < -2



Instruction for the animator	Text to be displayed in the working area (DT)
• The figure in blue has to appear then its label should appear. • Then the red figure has to appear. • After that the labeling of red figure has to appear. • In parallel to the fig. the text in DT has to appear. • After the figures, the 3, 4 lines in DT should appear.	• The signal *f(· ·)* is shown • The reversal and shifted version of *g(t) i.e., g(t-· · ·* is shown • Two functions do not overlap • Area under the product of the functions is zero

Step 4:

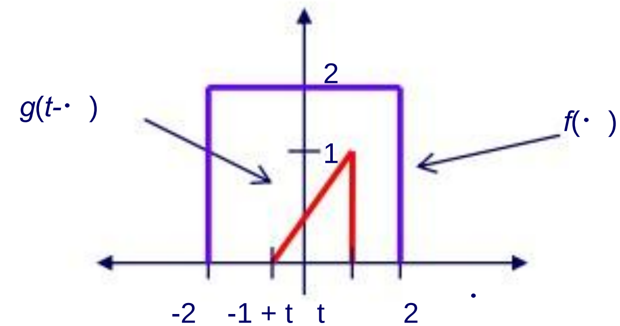
Stage - II : -2 ≤ t < -1



Instruction for the animator	Text to be displayed in the working area (DT)
• The figure in blue has to appear then its label should appear. • Then the red figure has to appear. • After that the labeling of red figure has to appear. • In parallel to the fig. the text in DT has to appear. • After the figures, the 3, 4 lines in DT should appear.	• The signal *f(· ·)* is shown • The reversal and shifted version of *g(t)* *i.e., g(t-· · ·* is shown • Part of *g(t-· · ·* overlaps part of *f(· ·)* • Area under the product 

Step 5:

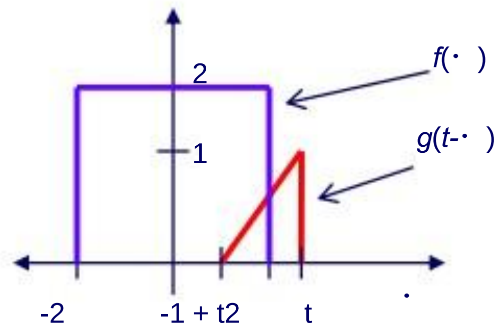
Stage - III : -1 ≤ t < 2



Instruction for the animator	Text to be displayed in the working area (DT)
• The figure in blue has to appear then its label should appear. • Then the red figure has to appear. • After that the labeling of red figure has to appear. • In parallel to the fig. the text in DT has to appear. • After the figures, the 3, 4 lines in DT should appear.	• The signal *f(• •)* is shown • The reversal and shifted version of *g(t)* *i.e., g(t-• • •* is shown • *g(t-• • •* completely overlaps *f(• •)* • Area under the product 

Step 6:

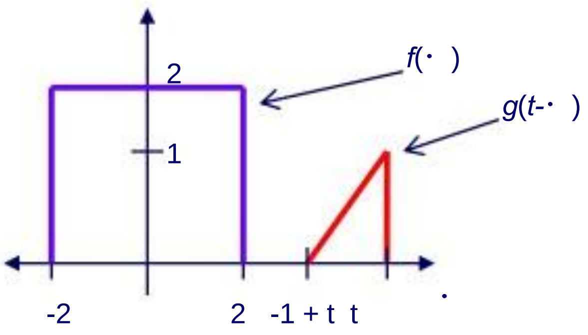
Stage - IV : 2 ≤ t < 3



Instruction for the animator	Text to be displayed in the working area (DT)
• The figure in blue has to appear then its label should appear. • Then the red figure has to appear. • After that the labeling of red figure has to appear. • In parallel to the fig. the text in DT has to appear. • After the figures, the 3, 4 lines in DT should appear.	• The signal *f(· ·)* is shown • The reversal and shifted version of *g(t)* *i.e., g(t-· · ·* is shown • Part of *g(t-· · ·* overlaps part of *f(· ·)* • Area under the product 

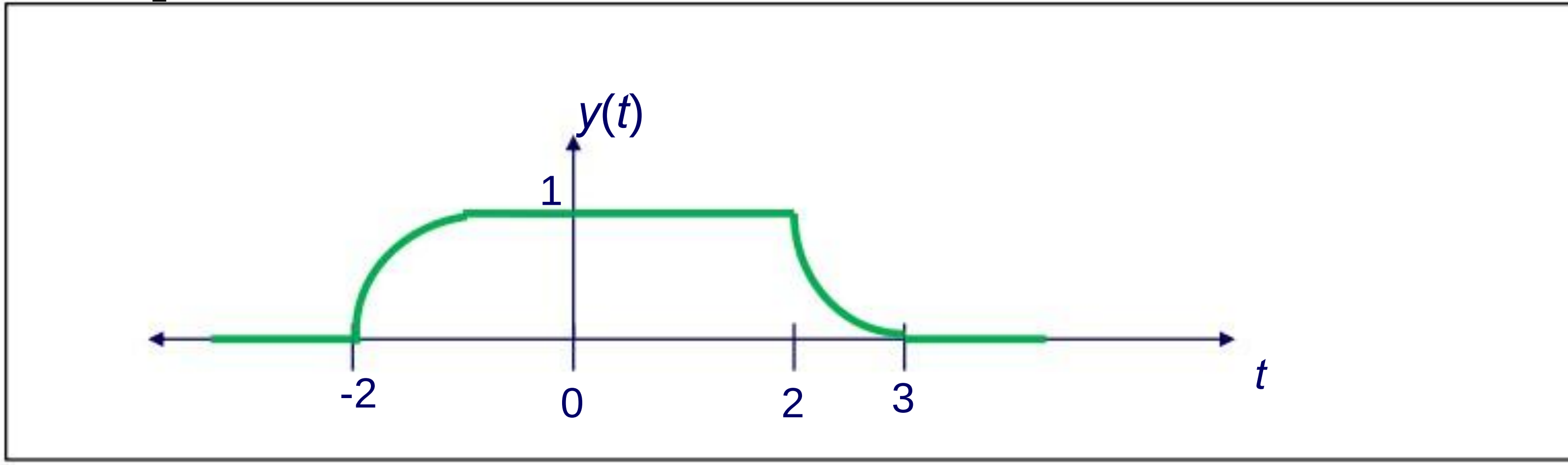
Step 7:

Stage - V : t ≥ 3

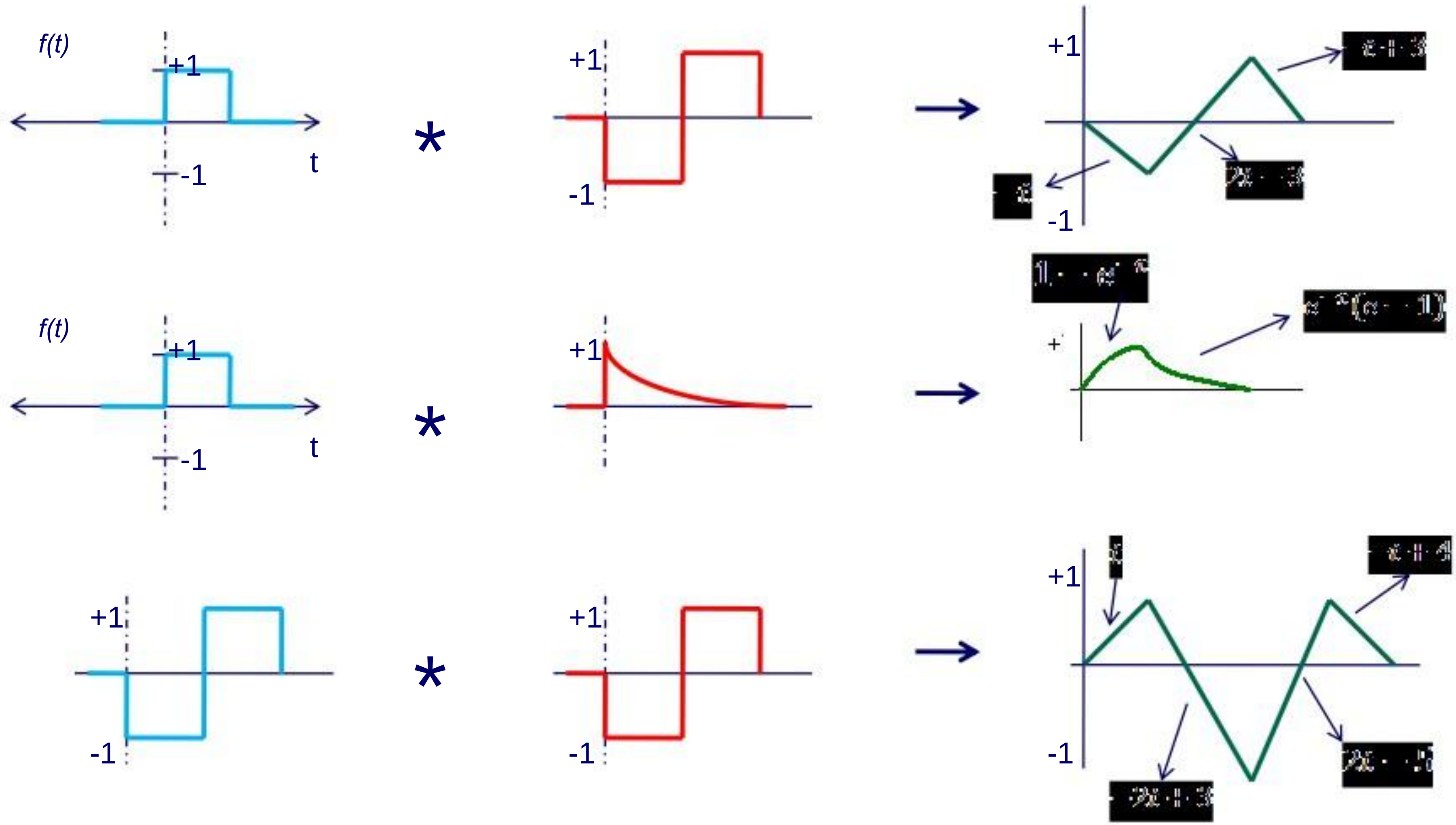


Instruction for the animator	Text to be displayed in the working area (DT)
• The figure in blue has to appear then its label should appear. • Then the red figure has to appear. • After that the labeling of red figure has to appear. • In parallel to the fig. the text in DT has to appear. • After the figures, the 3, 4 lines in DT should appear.	• The signal *f(· ·)* is shown • The reversal and shifted version of *g(t) i.e., g(t-· · ·* is shown • Two functions do not overlap • Area under the product of the functions is zero

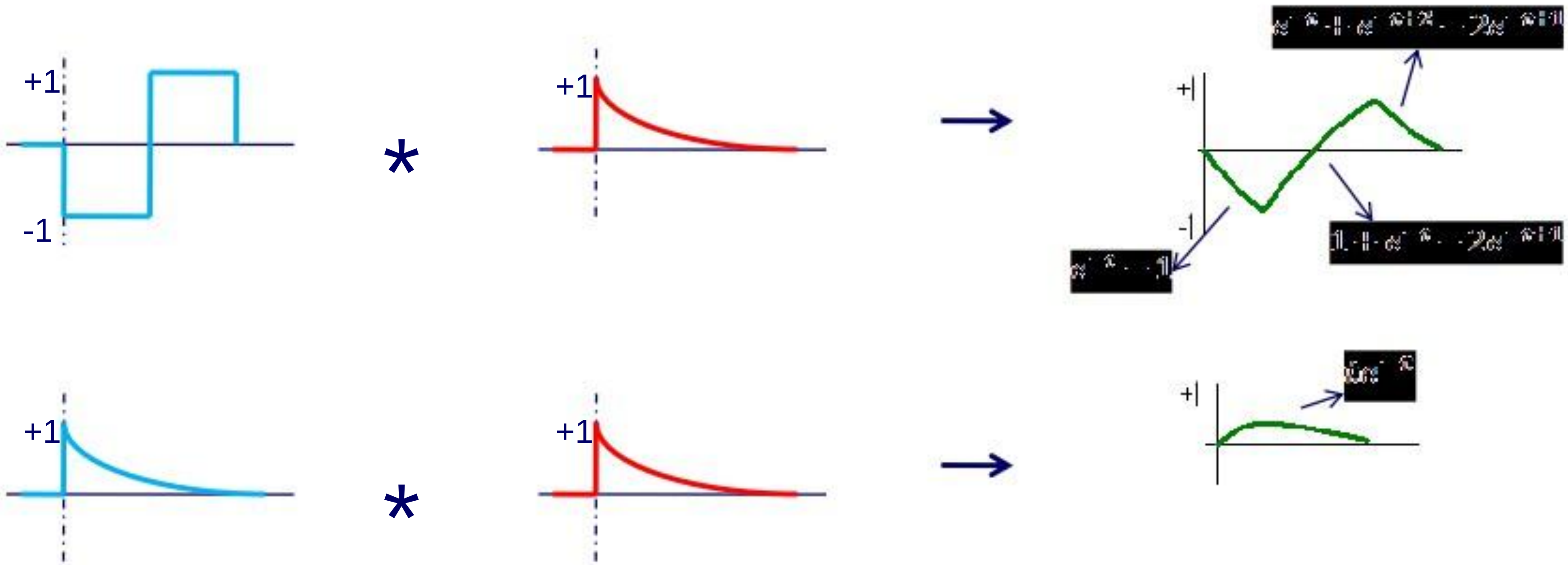
Step 8: Output of Convolution



Instruction for the animator	Text to be displayed in the working area (DT)
• The figure in green has to appear then its label should appear. • In parallel to the fig. the text in DT has to appear. • After the figure, the equations in DT should appear .	• The signal *y(t)* is shown $y(t) = f(t)*g(t) = \begin{cases} 0 & \text{for } t < -2 \\ -t^2 - 2t & \text{for } -2 \le t \le -1 \\ 1 & \text{for } -1 \le t \le 2 \\ t^2 - 6t + 9 & \text{for } 2 \le t \le 3 \\ 0 & \text{for } t > 3 \end{cases}$



f(t)
+1
-1
t
*
+1
-1
+1
-1
f(t)
+1
-1
t
*
+1
+1
-1
*
+1
-1
+1
-1


+1
-1
*
+1
+1
*
+1

Correlation and Auto-Correlation of Signals

Objectives

- Develop an intuitive understanding of the cross-correlation of two signals.
- Define the meaning of the auto-correlation of a signal.
- Develop a method to calculate the cross-correlation and auto-correlation of signals.
- Demonstrate the relationship between auto-correlation and signal power.
- Demonstrate how to detect periodicities in noisy signals using auto-correlation techniques.
- Demonstrate the application of cross-correlation to sonar or radar ranging

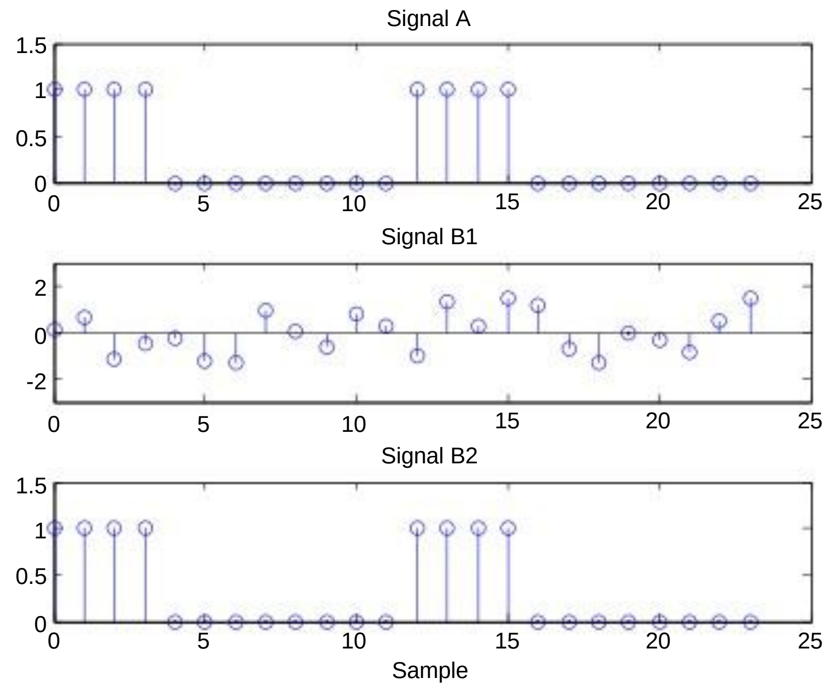
Correlation

- Correlation addresses the question: “to what degree is signal A similar to signal B.”
- An intuitive answer can be developed by comparing deterministic signals with stochastic signals.
  - Deterministic = a predictable signal equivalent to that produced by a mathematical function
  - Stochastic = an unpredictable signal equivalent to that produced by a random process

Three Signals

```
>> n=0:23;
>> A=[ones(1,4),zeros(1,8),ones(1,4),zeros(1,8)];
>> subplot (3,1,1),stem(n,A);axis([0 25 0 1.5]);title('Signal A')
>> B1=randn(size(A));  %The signal B1 is Gaussian noise with the same length as A
>> subplot(3,1,2),stem(n,B1);axis([0 25 -3 3]);title('Signal B1')
>> B2=A;
>> subplot(3,1,3),stem(n,B2); axis([0 25 0 1.5]);title('Signal B2');xlabel('Sample')
```

By inspection, A is "correlated" with B2, but B1 is "uncorrelated" with both A and B2. This is an intuitive and visual definition of "correlation."



Quantitative Correlation

- We seek a quantitative and algorithmic way of assessing correlation
- A possibility is to multiple signals sample-by-sample and average the results. This would give a relatively large positive value for identical signals and a near zero value for two random signals.

$$r_{12} = \frac{1}{N} \sum_{n=0}^{N-1} x_1[n] x_2[n]$$

Simple Cross-Correlation

- Taking the previous signals, A, B1(random), and B2 (identical to A):

```
>> A*B1'/length(A)
ans =
  -0.0047
>> A*B2'/length(A)
ans =
   0.3333
```

The small numerical result with A and B1 suggests those signals are uncorrelated while A and B2 are correlated.

Simple Cross-Correlation of Random Signals

```
>> n=0:100;
>> noise1=randn(size(n));
>> noise2=randn(size(n));
>> noise1*noise2'/length(noise1)
ans =
   0.0893
```

Are the two signals correlated?

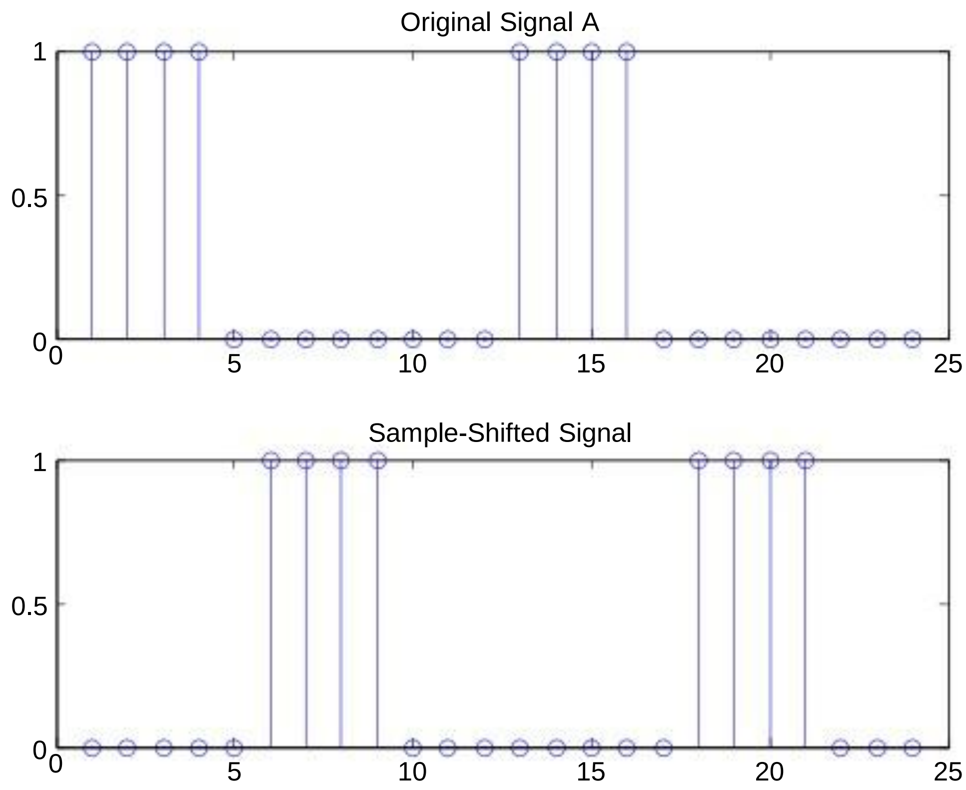
With high probability, the result is expected to be

$\leq \pm 2/\sqrt{N} = \pm 0.1990$

for two random (uncorrelated) signals

We would conclude these two signals are *uncorrelated*.

The Flaw in Simple Cross-Correlation



In this case, the simple cross-correlation would be zero despite the fact the two signals are obviously “correlated.”

Sample-Shifted Cross-Correlation

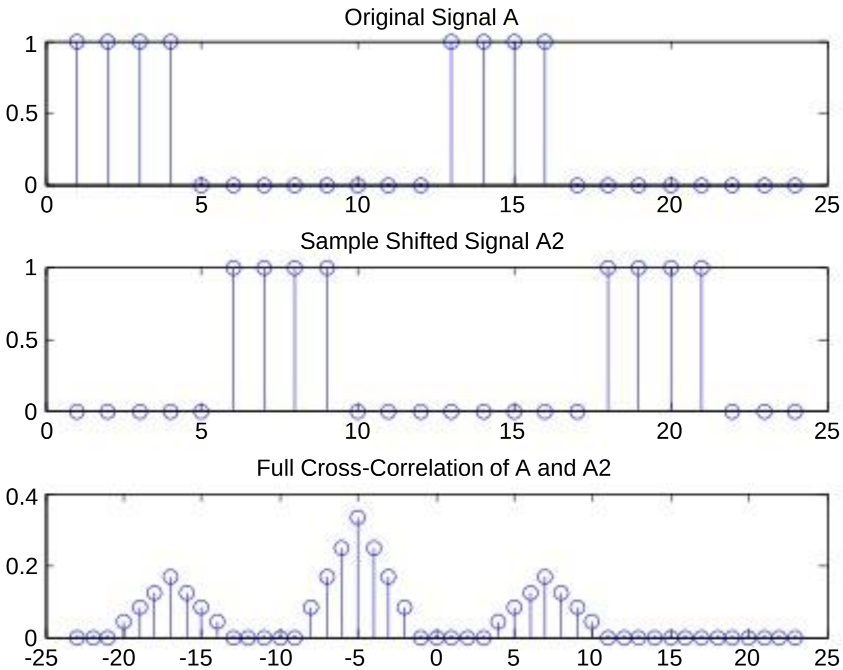
- Shift the signals k steps with respect to one another and calculate $r_{12}(k)$.
- All possible k shifts would produce a vector of values, the “full” cross-correlation.
- The process is performed in MATLAB by the command **xcorr**
- **xcorr** is equivalent to **conv** (convolution) with one of the signals taken in reverse order.

$$r_{12}(k) = \frac{1}{N} \sum_{n=0}^{N-1} x_1[n]\, x_2[n + k]$$

Full Cross-Correlation

```
>> A=[ones(1,4),zeros(1,8),ones(1,4),zeros(1,8)];
>> A2=filter([0,0,0,0,0,1],1,A);
>> [acor,lags]=xcorr(A,A2);
>> subplot(3,1,1),stem(A); title('Original Signal A')
>> subplot(3,1,2),stem(A2); title('Sample Shifted Signal A2')
>> subplot(3,1,3),stem(lags,acor/length(A)),title('Full Cross-Correlation of A and A2')
```

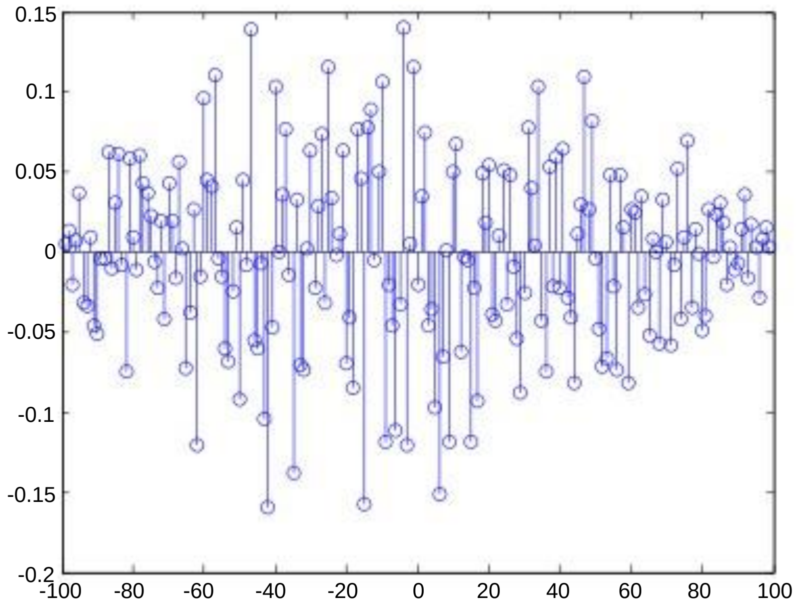
Signal A2 shifted to the left by 5 steps makes the signals identical and $r_{12} = 0.333$



Full Cross-Correlation of Two Random Signals

```
>> N=1:100;
>> n1=randn(size(N));
>> n2=randn(size(N));
>> [acor,lags]=xcorr(n1,n2);
>> stem(lags,acor/length(n1));
```

The cross-correlation is random and shows no peak, which implies no correlation



Auto-Correlation

- The cross-correlation of a signal with itself is called the *auto-correlation*

$$r_{11}(k) = \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\,x_1[n+k]$$

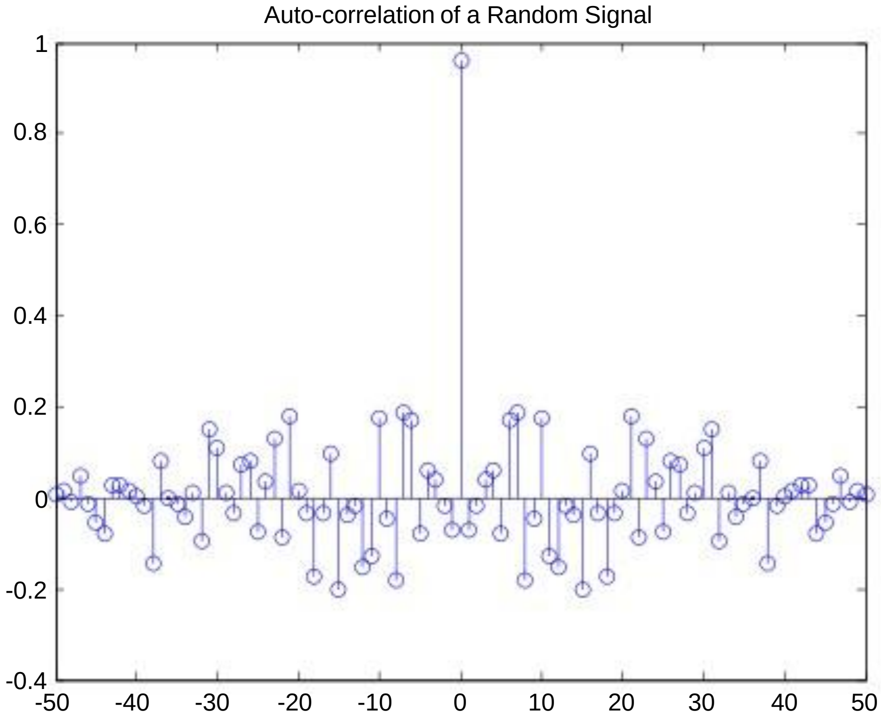
- The "zero-lag" auto-correlation is the same as the mean-square *signal power*.

$$r_{11}(0) = \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\,x_1[n] = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$$

Auto-Correlation of a Random Signal

```
>> n=0:50;
>> N=randn(size(n));
>> [rNN,k]=xcorr(N,N);
>> stem(k,rNN/length(N));title('Auto-correlation of a Random Signal')
```

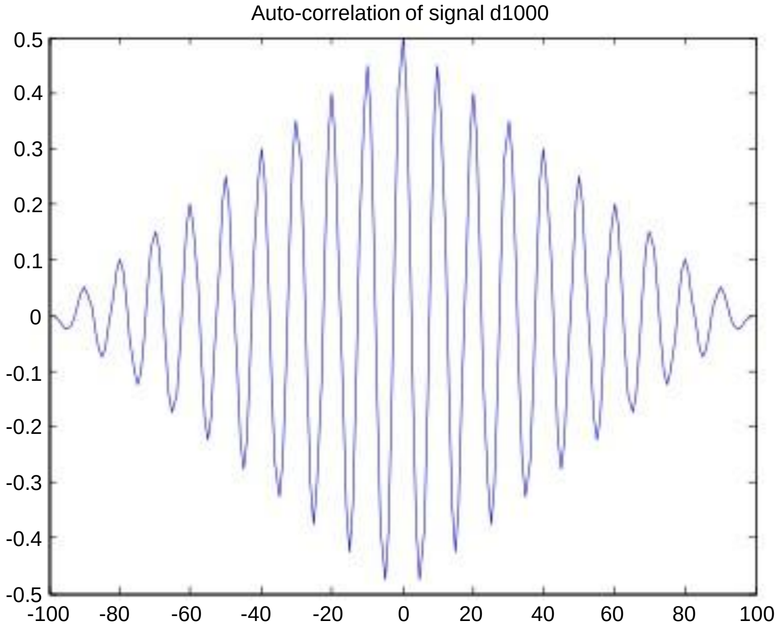
Mathematically, the auto-correlation of a random signal is like the impulse function



Auto-Correlation of a Sinusoid

```
>> n=0:99;
>> omega=2*pi*100/1000;
>> d1000=sin(omega*n);
>> [acor_d1000,k]=xcorr(d1000,d1000);
>> plot(k,acor_d1000/length(d1000));
>> title('Auto-correlation of signal d1000')
```

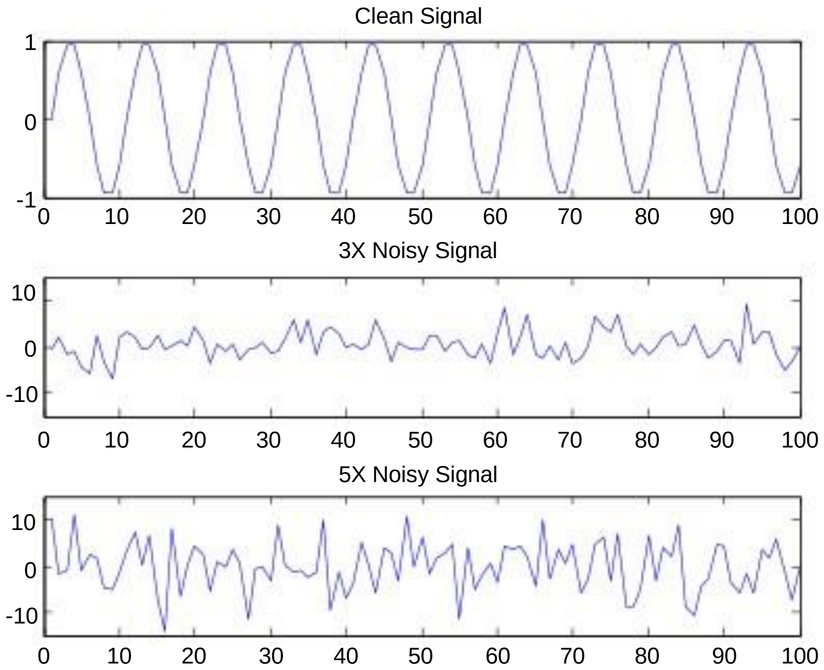
The auto-correlation vector has the same frequency components as the original signal



Identifying a Sinusoidal Signal Masked by Noise

```
>> n=0:1999;
>> omega=2*pi*100/1000;
>> d=sin(omega*n);
>> d3n=d+3*randn(size(d));  % The sinusoid is contaminated with 3X noise
>> d5n=d+5*randn(size(d));  % The sinusoid is contaminated with 5X noise.
>> subplot(3,1,1),plot(d(1:100)),title('Clean Signal')
>> subplot(3,1,2),plot(d3n(1:100)),title('3X Noisy Signal'), axis([0,100,-15,15])
>> subplot(3,1,3),plot(d5n(1:100)),title('5X Noisy Signal'), axis([0,100,-15,15])
```

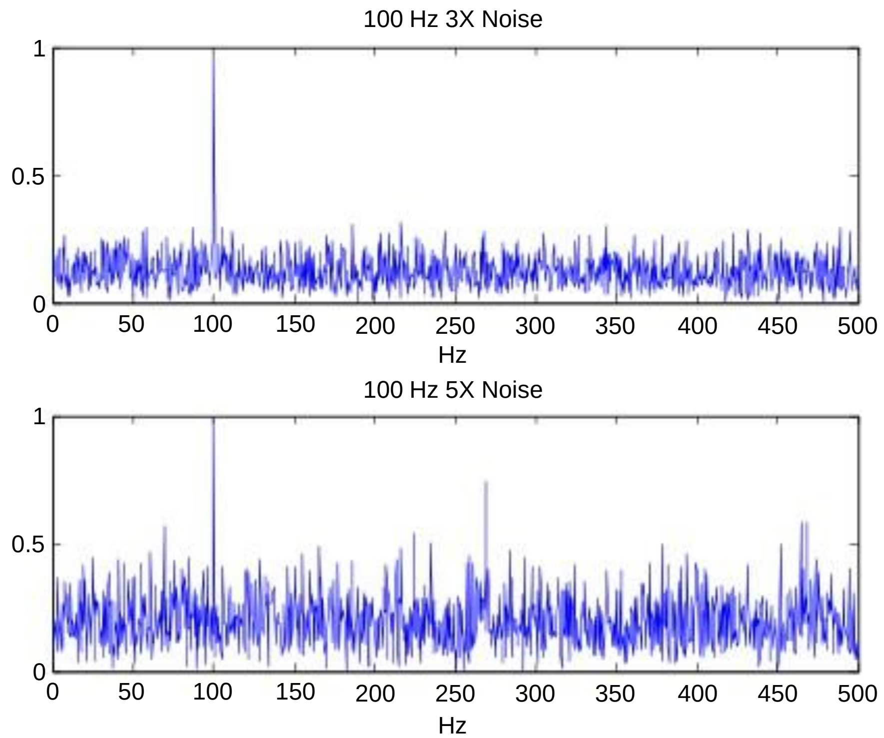
It is very difficult to "see" the sinusoid in the noisy signals



Identifying a Sinusoidal Signal Masked by Noise (Normal Spectra)

```
>> n=0:1999;
>> omega=2*pi*100/1000;
>> d=sin(omega*n);
>> d3n=d+3*randn(size(d));  % The sinusoid is contaminated with 3X noise
>> d5n=d+5*randn(size(d));  % The sinusoid is contaminated with 5X noise.
>> subplot(2,1,1),fft_plot(d3n,1000);title('100 Hz 3X Noise')
>> subplot(2,1,2),fft_plot(d5n,1000);title('100 Hz 5X Noise')
```

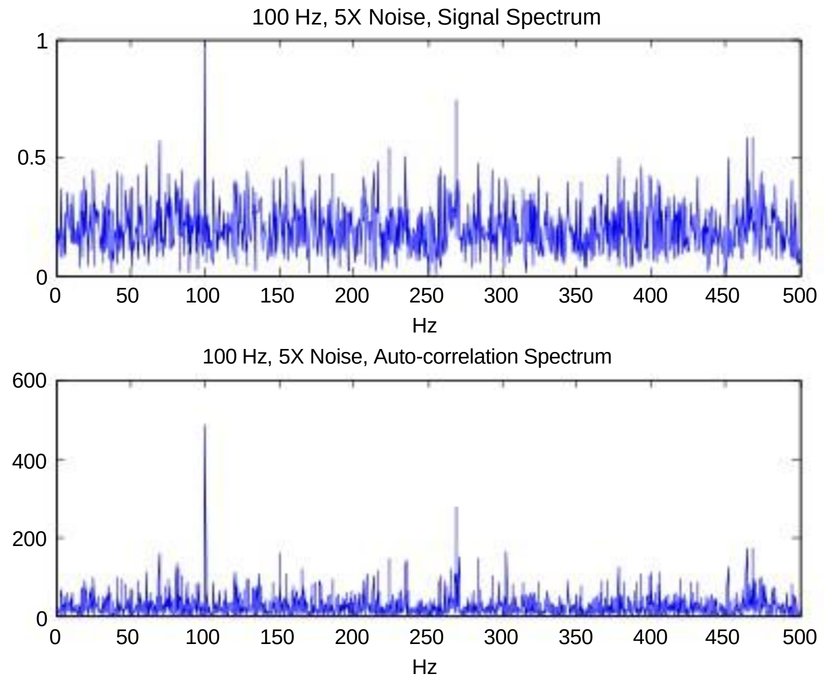
Normal spectra of a sinusoid masked by noise: High noise power makes detection less certain



Identifying a Sinusoidal Signal Masked by Noise (Auto-correlation Spectra)

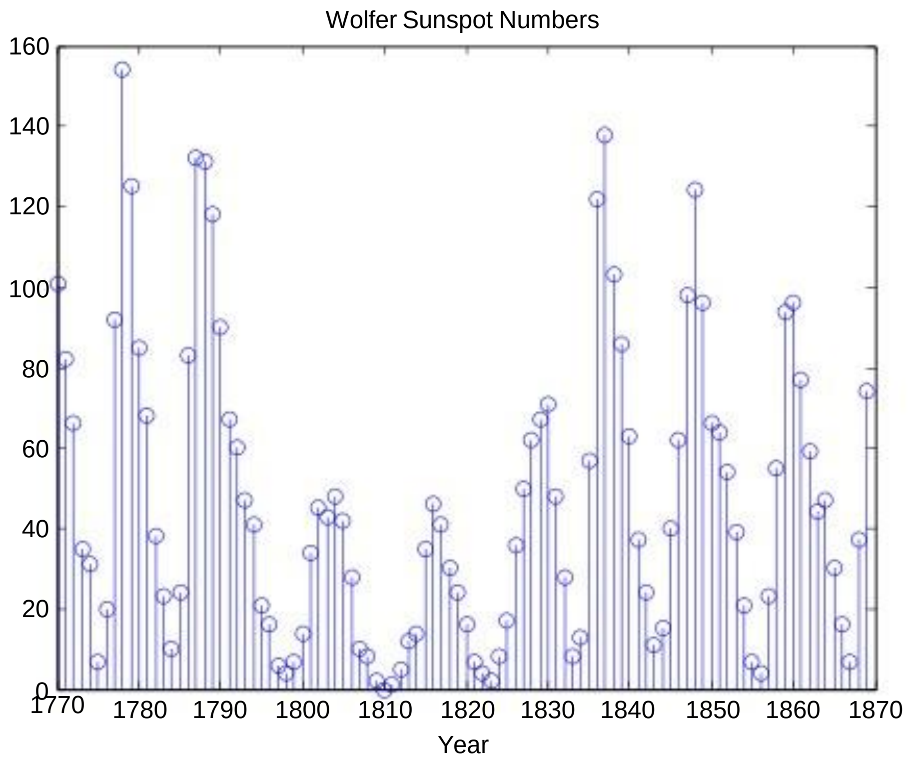
```
>> acor3n=xcorr(d3n,d3n);
>> acor5n=xcorr(d5n,d5n);
>> subplot(2,1,1),fft_plot(d3n,1000);title('100 Hz, 3X Noise, Signal Spectrum')
>> subplot(2,1,2),fft_plot(acor3n,1000);title('100 Hz, 3X Noise, Auto-correlation Spectrum')
>> figure, subplot(2,1,1),fft_plot(d5n,1000);title('100 Hz, 5X Noise, Signal Spectrum')
>> subplot(2,1,2),fft_plot(acor5n,1000);title('100 Hz, 5X Noise, Auto-correlation Spectrum')
```

The *auto-correlation* of a noisy signal provides greater S/N in detecting dominant frequency components compared to a normal FFT



Detecting Periodicities in Noisy Data: Annual Sunspot Data

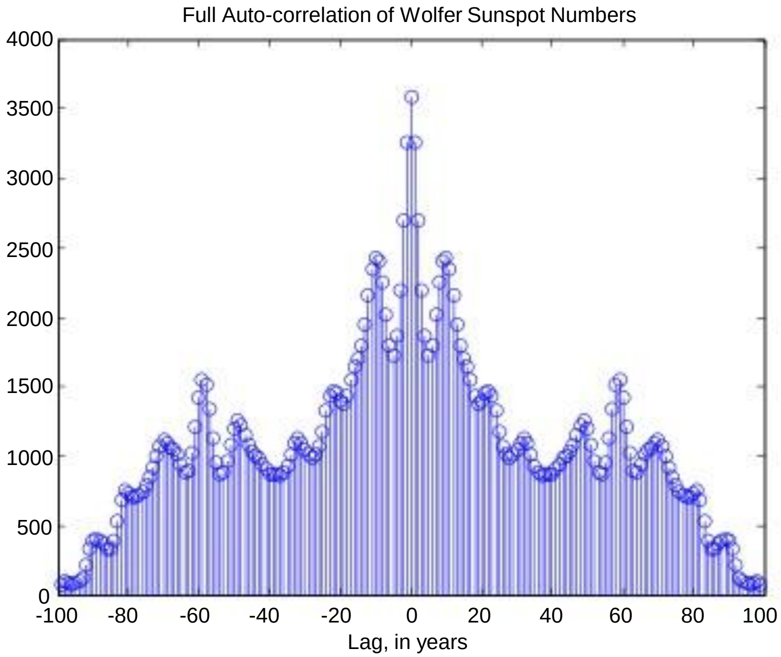
```
>> load wolfer_numbers
>> year=sunspots(:,1);
>> spots=sunspots(:,2);
>> stem(year,spots);title('Wolfer Sunspot Numbers');xlabel('Year')
```

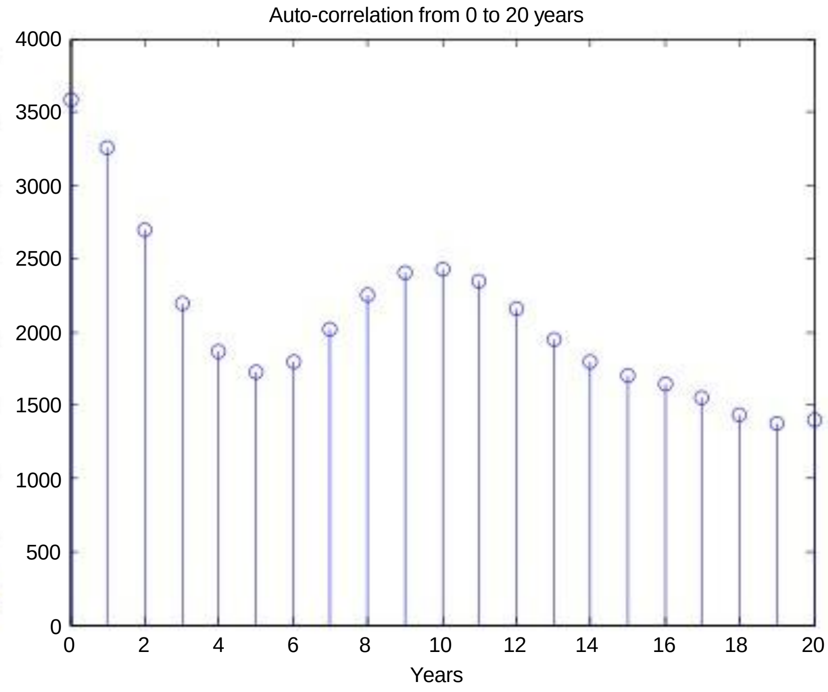


Detecting Periodicities in Noisy Data: Annual Sunspot Data

```
>> [acor,lag]=xcorr(spots);
>> stem(lag,acor/length(spots));
>> title('Full Auto-correlation of Wolfer Sunspot Numbers')
>> xlabel('Lag, in years')
>> figure, stem(lag(100:120),acor(100:120)/length(spots));
>> title('Auto-correlation from 0 to 20 years')
>> xlabel('Years')
```

Autocorrelation has detected a periodicity of 9 to 11 years

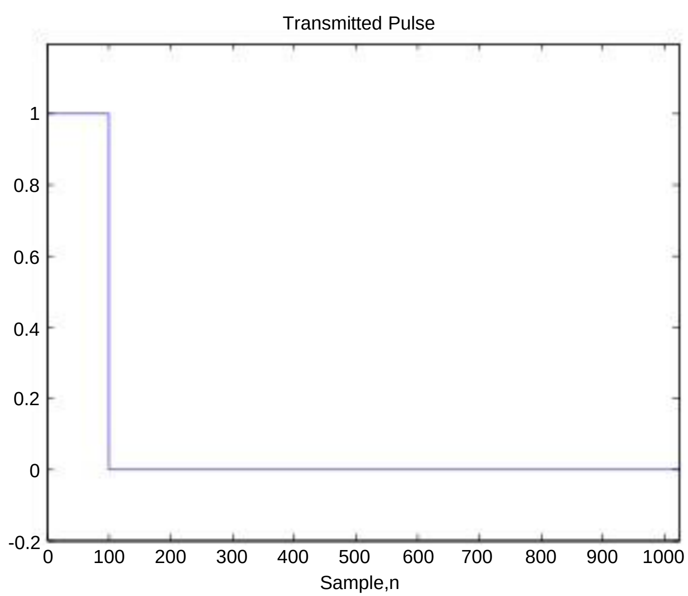


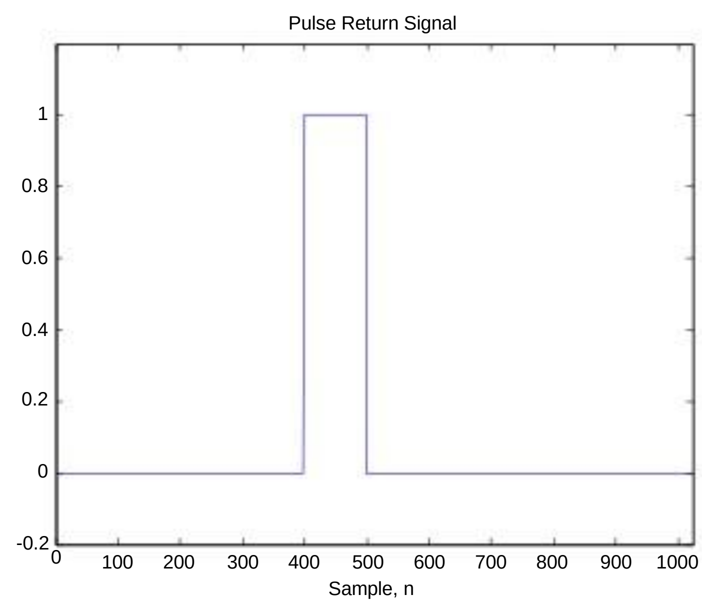


Sonar and Radar Ranging

```
>> x=[ones(1,100),zeros(1,924)];
>> n=0:1023;
>> plot(n,x); axis([0 1023 -.2, 1.2])
>> title('Transmitted Pulse');xlabel('Sample,n')
>> h=[zeros(1,399),1];            % Impulse response for z-400 delay
>> x_return=filter(h,1,x);        % Put signal thru delay filter
>> figure,plot(n,x_return); axis([0 1023 -.2, 1.2])
>> title('Pulse Return Signal');xlabel('Sample, n')
```

Simulation of a transmitted and received pulse (echo) with a 400 sample delay

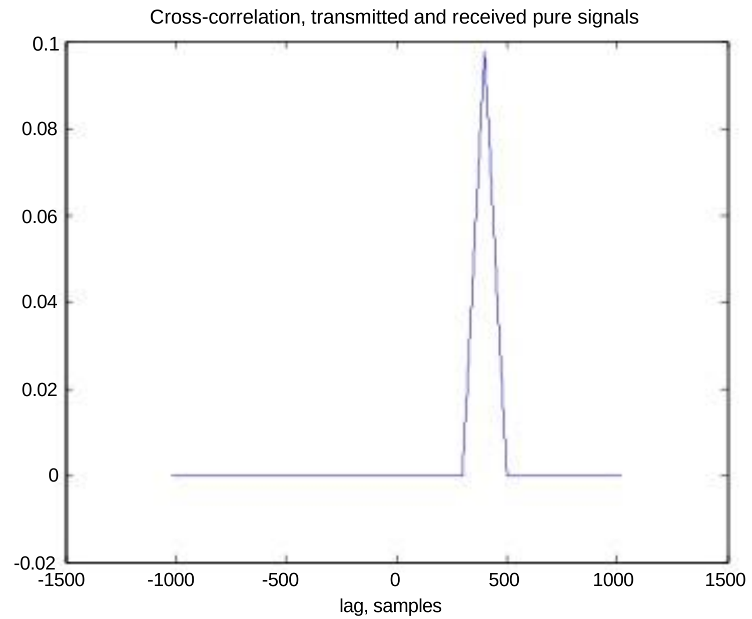




Sonar and Radar Ranging

```
>> [xcor_pure,lags]=xcorr(x_return,x);
>> plot(lags,xcor_pure/length(x))
>> title('Cross-correlation, transmitted and received pure signals')
>> xlabel('lag, samples')
```

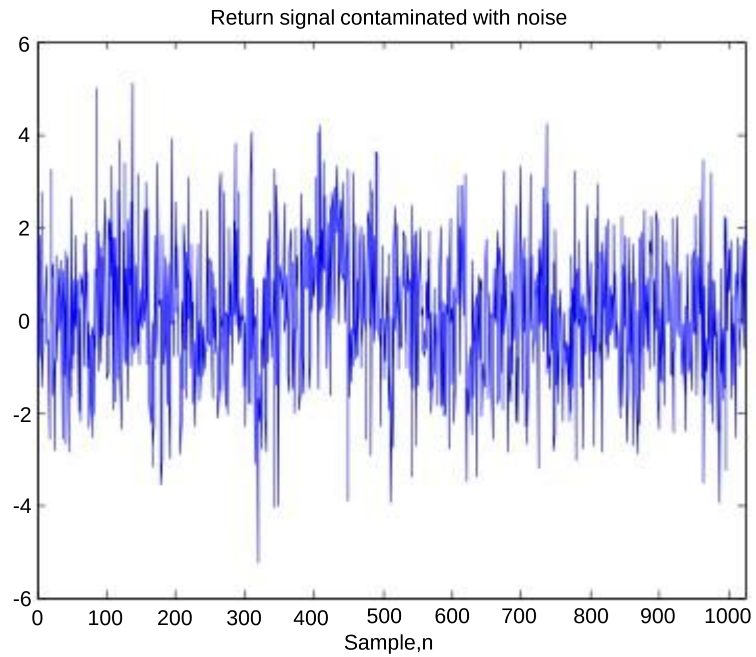
The cross-correlation of the transmitted and received signals shows they are correlated with a 400 sample delay



Sonar and Radar Ranging

```
>> x_ret_n=x_return+1.5*randn(size(x_return));
>> plot(n,x_ret_n); axis([0 1023 -6, 6])          %Note change in axis range
>> title('Return signal contaminated with noise')
>> xlabel('Sample,n')
```

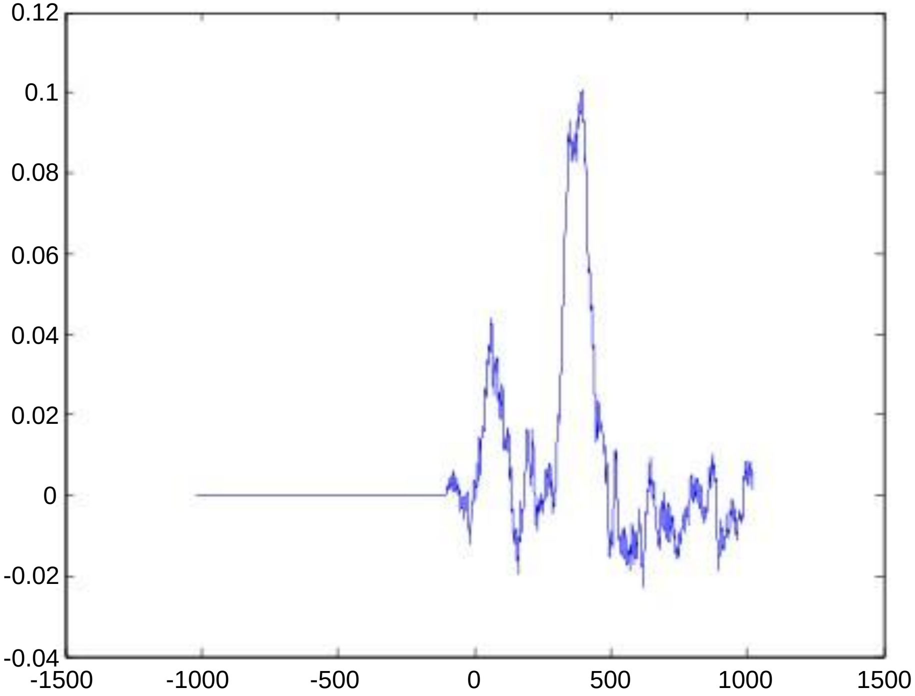
The presence of the return signal in the presence of noise is almost impossible to see



Sonar and Radar Ranging

```
>> [xcor,lags]=xcorr(x_ret_n,x);
>> plot(lags,xcor/length(x))
```

Cross-correlation of the transmitted signal with the noisy echo clearly shows a correlation at a delay of 400 samples



Summary

- Cross-correlation allows assessment of the degree of similarity between two signals.
  - Its application to identifying a sonar/radar return echo in heavy noise was illustrated.
- Auto-correlation (the correlation of a signal with itself) helps identify signal features buried in noise.

The Laplace Transform

Generalizing the Fourier Transform

The CTFT expresses a time-domain signal as a linear combination of **complex sinusoids** of the form $e^{j\omega t}$. In the generalization of the CTFT to the Laplace transform, the complex sinusoids become **complex exponentials** of the form e^{st} where s can have any complex value. Replacing the complex sinusoids with complex exponentials leads to this definition of the Laplace transform.

$$\mathcal{L}\left(\mathrm{x}(t)\right) = \mathrm{X}(s) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-st}\, dt$$

$$\mathrm{x}(t) \xleftrightarrow{\mathcal{L}} \mathrm{X}(s)$$

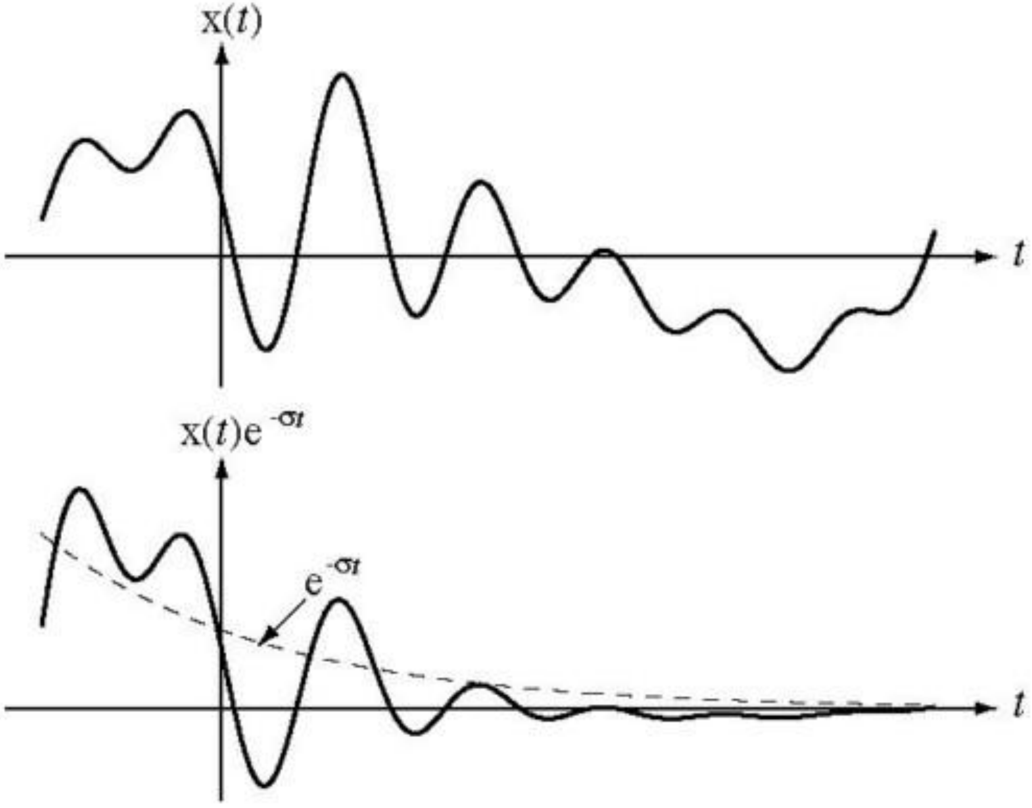
Generalizing the Fourier Transform

The variable s is viewed as a generalization of the variable ω of the form $s = \sigma + j\omega$. Then, when σ, the real part of s, is zero, the Laplace transform reduces to the CTFT. Using $s = \sigma + j\omega$ the Laplace transform is

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} \, dt$$

$$= \mathcal{F}\left[x(t) e^{-\sigma t} \right]$$

which is the Fourier transform of $x(t) e^{-\sigma t}$



Generalizing the Fourier Transform

e extra factor $e^{-\sigma t}$ is sometimes called a **convergence factor** because, when chosen properly, it makes the integral converge for some signals for which it would not otherwise converge. For example, strictly speaking, the signal $A\,\mathrm{u}(t)$ does not have a CTFT because the integral does not converge. But if it is multiplied by the convergence factor, and the real part of s is chosen appropriately, the CTFT integral will converge.

$$\int_{-\infty}^{\infty} A\mathrm{u}(t)e^{-j\omega t}\,dt = A\int_{0}^{\infty} e^{-j\omega t}\,dt \leftarrow \text{Does not converge}$$

$$\int_{-\infty}^{\infty} Ae^{-\sigma t}\,\mathrm{u}(t)e^{-j\omega t}\,dt = A\int_{0}^{\infty} e^{-(\sigma+j\omega)t}\,dt \leftarrow \text{Converges (if } \sigma > 0\text{)}$$

Complex Exponential Excitation

If a continuous-time LTI system is excited by a complex exponential $\mathrm{x}(t) = Ae^{st}$, where A and s can each be any complex number, the system response is also a complex exponential of the same functional form except multiplied by a complex constant. The response is the convolution of the excitation with the impulse response and that is

$$\mathrm{y}(t) = \int_{-\infty}^{\infty} \mathrm{h}(\tau)\mathrm{x}(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \mathrm{h}(\tau) Ae^{s(t-\tau)}\,d\tau = \underbrace{Ae^{st}}_{\mathrm{x}(t)} \int_{-\infty}^{\infty} \mathrm{h}(\tau)e^{-s\tau}\,d\tau$$

The quantity $\mathrm{H}(s) = \int_{-\infty}^{\infty} \mathrm{h}(\tau)e^{-s\tau}\,d\tau$ is called the **Laplace transform** of $\mathrm{h}(t)$.

Complex Exponential Excitation

Let $\mathrm{x}(t) = \underbrace{(6 + j3)}_{A} e^{\overbrace{(3-j2)}^{s}t} = (6.708\angle 0.4637)e^{(3-j2)t}$

and let $\mathrm{h}(t) = e^{-4t}\,\mathrm{u}(t)$. Then $\mathrm{H}(s) = \dfrac{1}{s+4}$, $\sigma > -4$ and,

in this case, $s = 3 - j2 = \sigma + j\omega$ with $\sigma = 3 > -4$ and $\omega = -2$.

$$\mathrm{y}(t) = \mathrm{x}(t)\mathrm{H}(s) = \frac{6 + j3}{3 - j2 + 4} e^{(3-j2)t} = (0.6793\angle 0.742)e^{(3-j2)t}.$$

The response is the same functional form as the excitation but multiplied by a different complex constant. This only happens when the excitation is a complex exponential and that is what makes complex exponentials unique.

Pierre-Simon Laplace

3/23/1749 - 3/2/1827

The Transfer Function

e $x(t)$ be the excitation and let $y(t)$ be the response of a system with impulse response $h(t)$. The Laplace transform of $y(t)$ is

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt = \int_{-\infty}^{\infty} \left[h(t)*x(t)\right]e^{-st}dt$$

$$Y(s) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right)e^{-st}dt$$

$$Y(s) = \int_{-\infty}^{\infty} h(\tau)d\tau \int_{-\infty}^{\infty} x(t-\tau)e^{-st}dt$$

The Transfer Function

Let $x(t) = u(t)$ and let $h(t) = e^{-4t}u(t)$. Find $y(t)$.

$$y(t) \qquad \int_{-\infty}^{\infty} \qquad \int_{-\infty}^{\infty} (\quad) u(t-\tau)\,d\tau$$

$$y(t) \begin{cases} \int_0^t (\quad)\,d\tau = e^{-4t}\int_0^t e^{4\tau}\,d\tau = e^{-4t}\dfrac{e^{4t}-1}{4} = \dfrac{1-e^{-4t}}{4}, & t > 0 \\ 0, & t < 0 \end{cases}$$

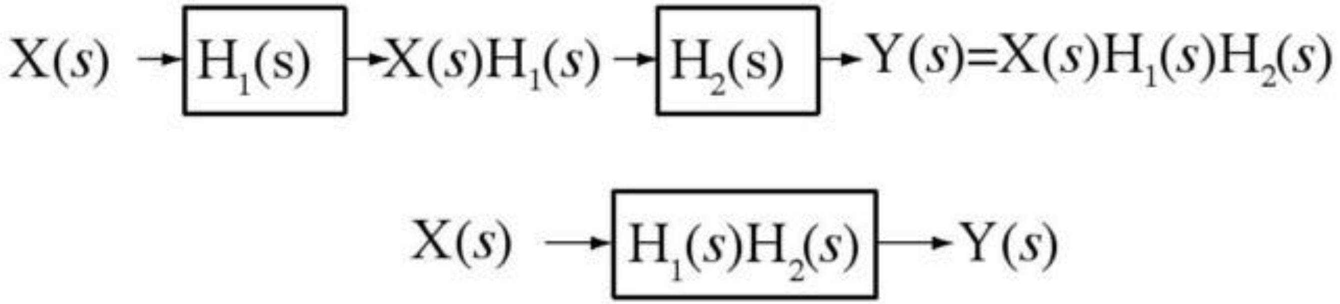
$$y(t) \qquad (\qquad) u(t)$$

$$X(s) = 1/s,\ H(s) = \frac{1}{s+4} \Rightarrow Y(s) = \frac{1}{s} \times \frac{1}{s+4} = \frac{1/4}{s} - \frac{1/4}{s+4}$$

$$(\qquad) u(t)$$

Cascade-Connected Systems

If two systems are cascade connected the transfer function of the overall system is the product of the transfer functions of the two individual systems.



Direct Form II Realization

A very common form of transfer function is a ratio of two polynomials in s,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{N} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

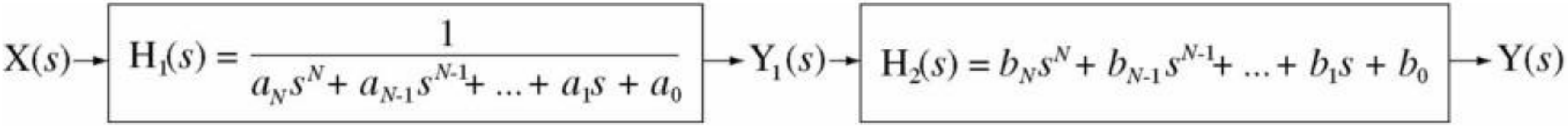
Direct Form II Realization

The transfer function can be conceived as the product of two transfer functions,

$$H_1(s) = \frac{Y_1(s)}{X(s)} = \frac{1}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

and

$$H_2(s) = \frac{Y(s)}{Y_1(s)} = b_N s^N + b_{N-1} s^{N-1} + \cdots + b_1 s + b_0$$



Direct Form II Realization

From

$$\mathrm{H}_1(s) = \frac{\mathrm{Y}_1(s)}{\mathrm{X}(s)} = \frac{1}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

we get

$$\mathrm{X}(s) = \left[a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0\right] \mathrm{Y}_1(s)$$

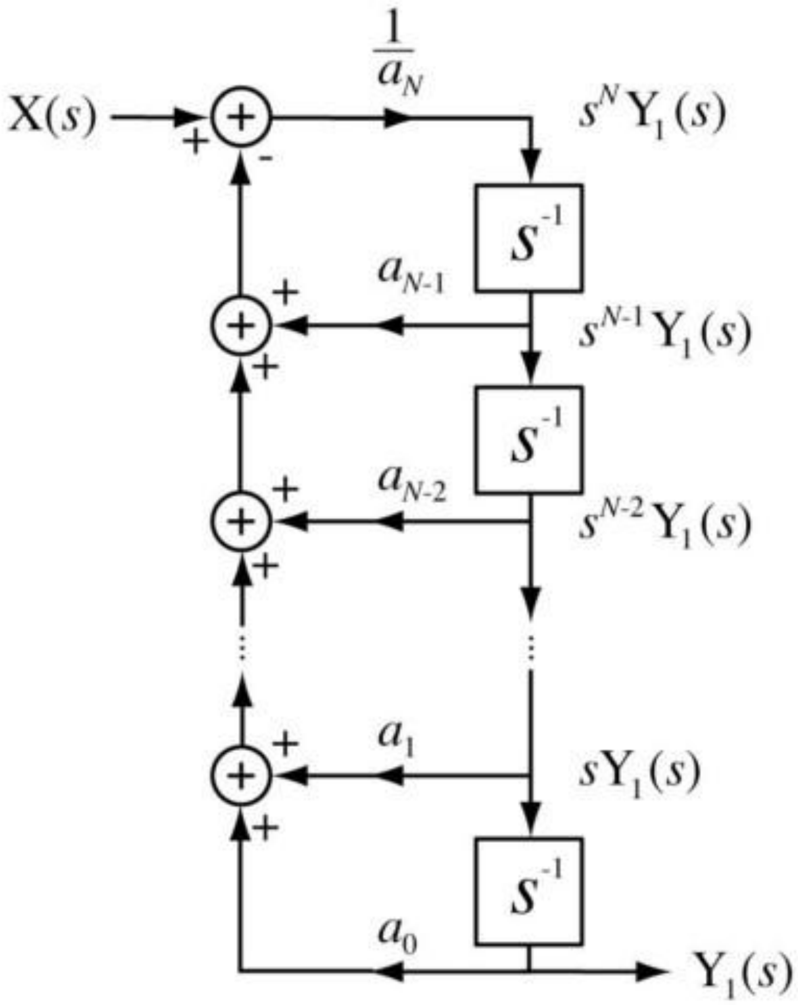
or

$$\mathrm{X}(s) = a_N s^N \, \mathrm{Y}_1(s) + a_{N-1} s^{N-1} \, \mathrm{Y}_1(s) + \cdots + a_1 s \, \mathrm{Y}_1(s) + a_0 \, \mathrm{Y}_1(s)$$

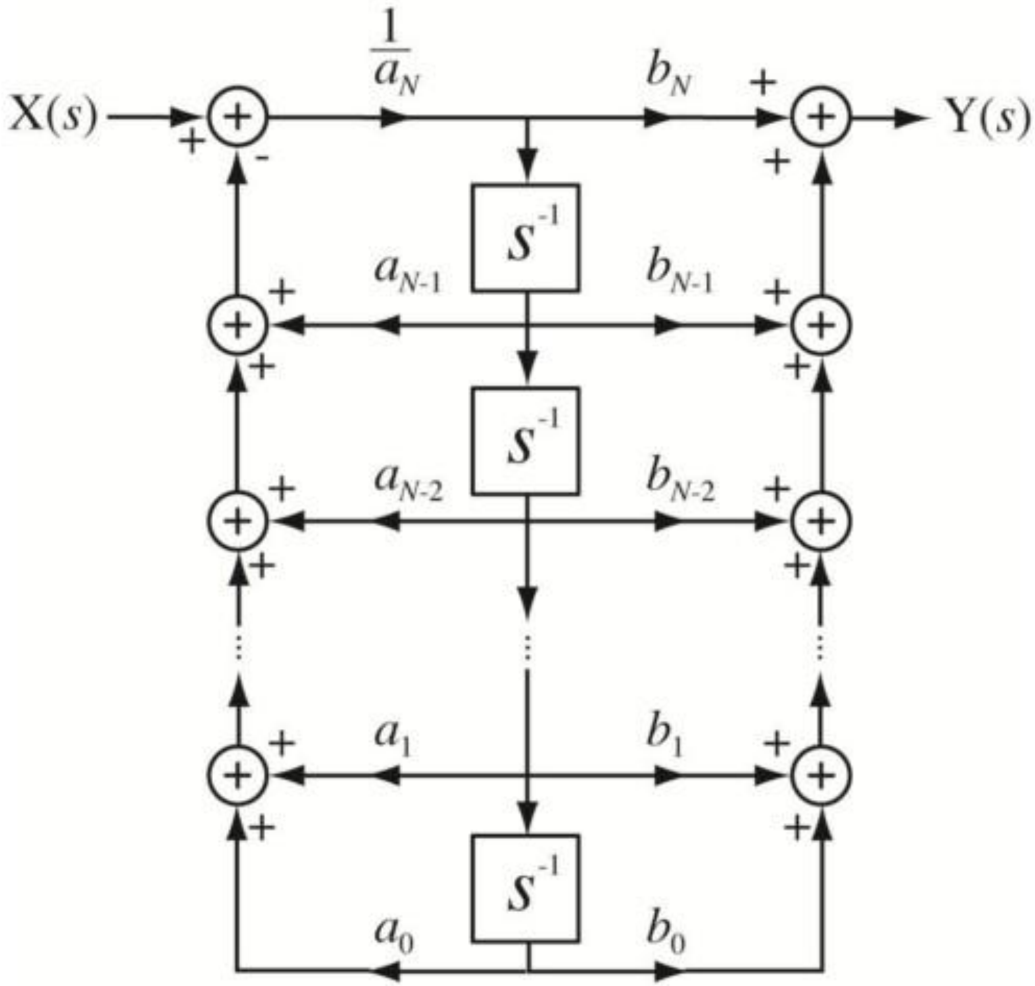
Rearranging

$$s^N \, \mathrm{Y}_1(s) = \frac{1}{a_N}\left\{\mathrm{X}(s) - \left[a_{N-1} s^{N-1} \, \mathrm{Y}_1(s) + \cdots + a_1 s \, \mathrm{Y}_1(s) + a_0 \, \mathrm{Y}_1(s)\right]\right\}$$

Direct Form II Realization



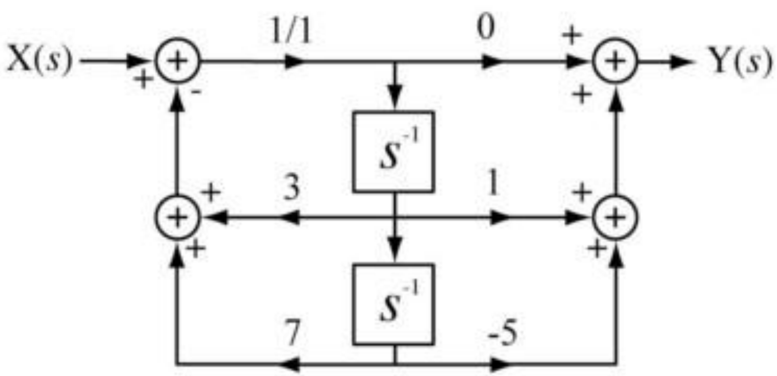
Direct Form II Realization



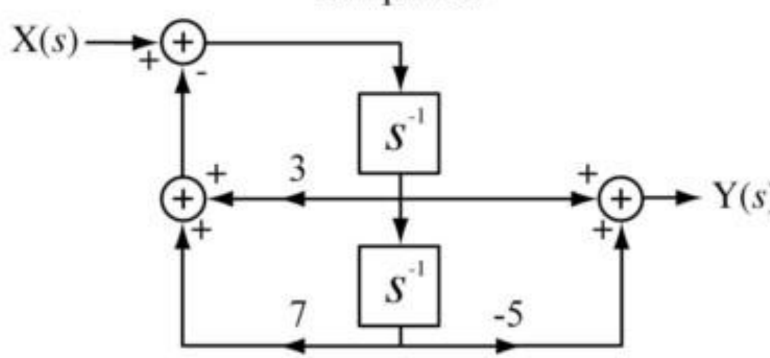
Direct Form II Realization

A system is defined by y¢¢(t) + 3y¢(t) + 7 y(t) = x¢(t) - 5 x(t).

$$H(s) = \frac{s-5}{s^2 + 3s + 7}$$



Simplified



Inverse Laplace Transform

There is an inversion integral

$$y(t) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} Y(s)e^{st}ds \quad , \quad s = \sigma + j\omega$$

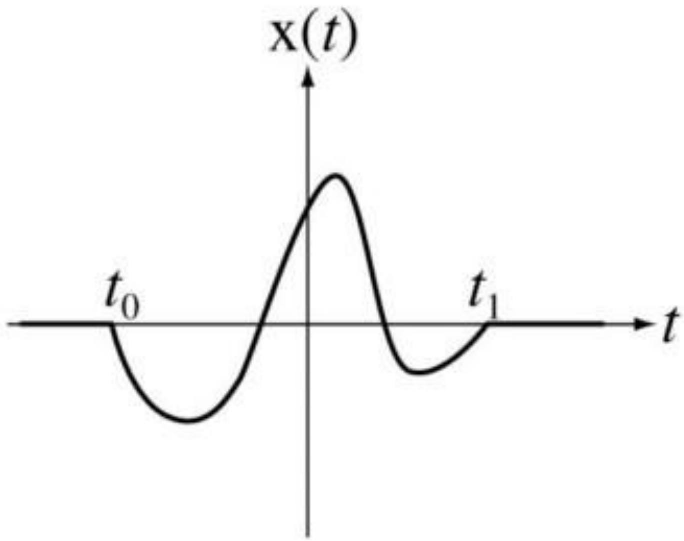
for finding $y(t)$ from $Y(s)$, but it is rarely used in practice.

Usually inverse Laplace transforms are found by using tables of standard functions and the properties of the Laplace transform.

Existence of the Laplace Transform

Time Limited Signals

If $x(t) = 0$ for $t < t_0$ and $t > t_1$ it is a **time limited** signal. If $x(t)$ is also bounded for all t, the Laplace transform integral converges and the Laplace transform exists for all s.



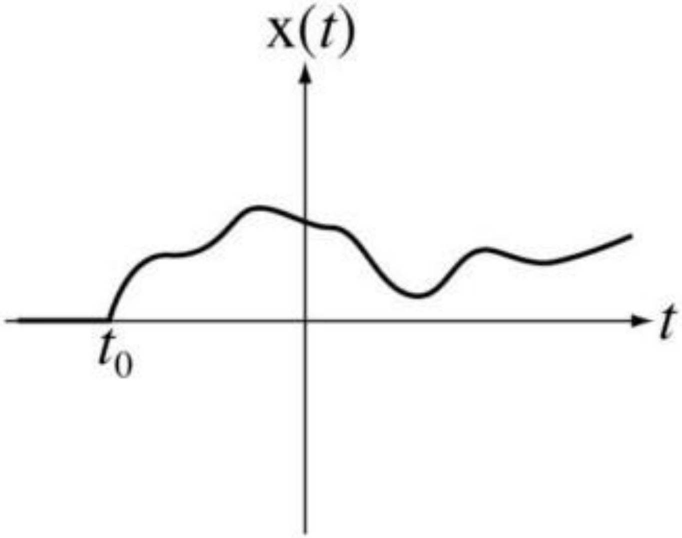
Existence of the Laplace Transform

Let $x(t) = \mathrm{rect}(t) = u(t+1/2) - u(t-1/2)$.

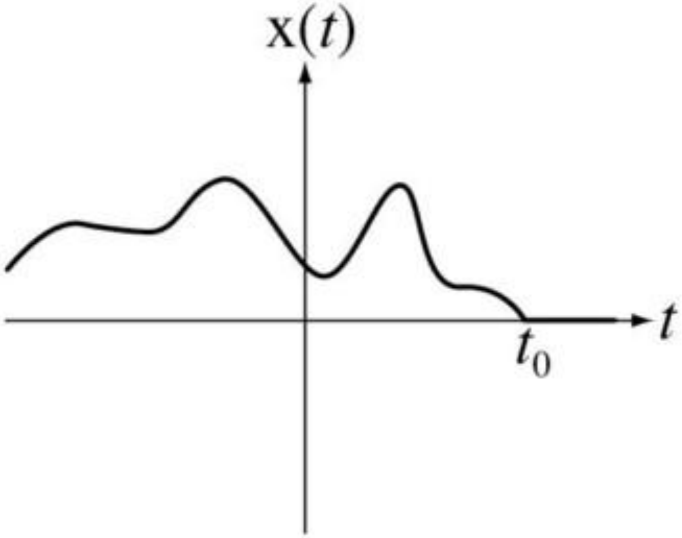
$$X(s) = \int_{-\infty}^{\infty} \mathrm{rect}(t) e^{-st}\, dt = \int_{-1/2}^{1/2} e^{-st}\, dt = \frac{e^{-s/2} - e^{s/2}}{-s} = \frac{e^{s/2} - e^{-s/2}}{s}, \text{ All } s$$

Existence of the Laplace Transform

Right- and Left-Sided Signals



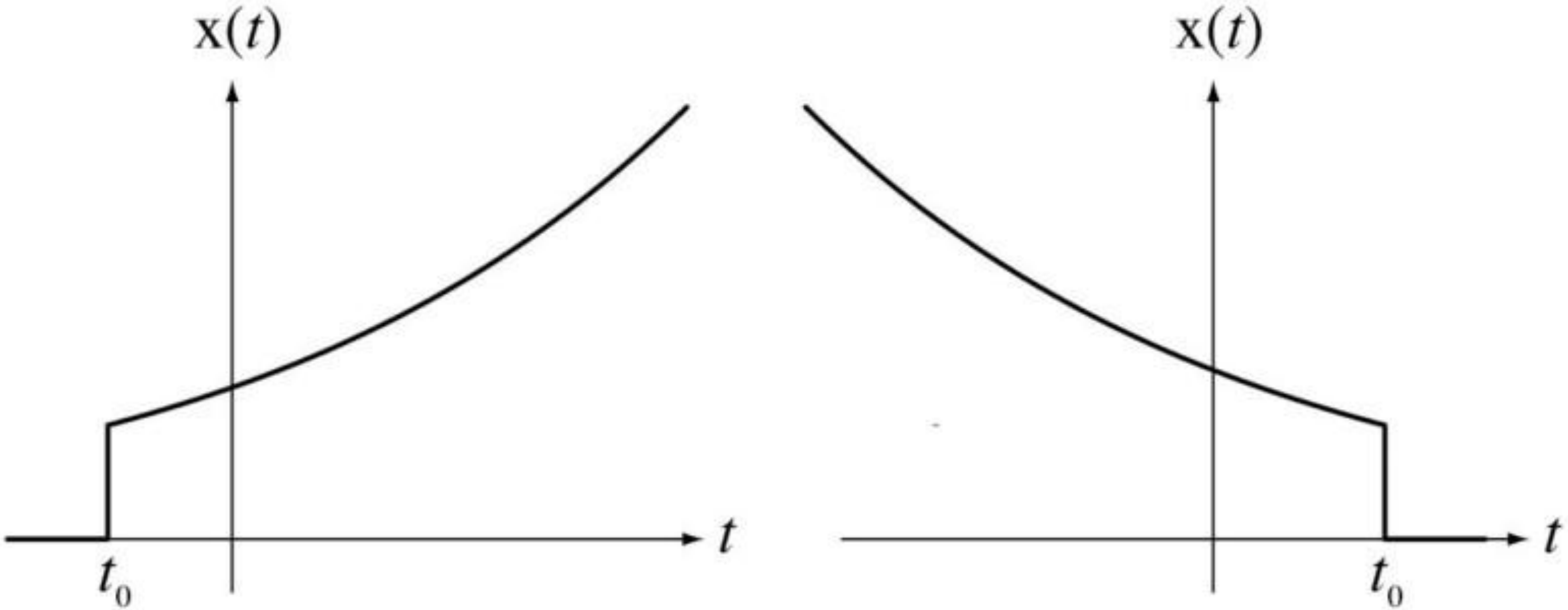
Right-Sided



Left-Sided

Existence of the Laplace Transform

Right- and Left-Sided Exponentials



Right-Sided

Left-Sided

$x(t) = e^{at}\,u(t - t_0)$, a Î·

$x(t) = e^{bt}\,u(t_0 - t)$, b Î·

Existence of the Laplace Transform

Right-Sided Exponential

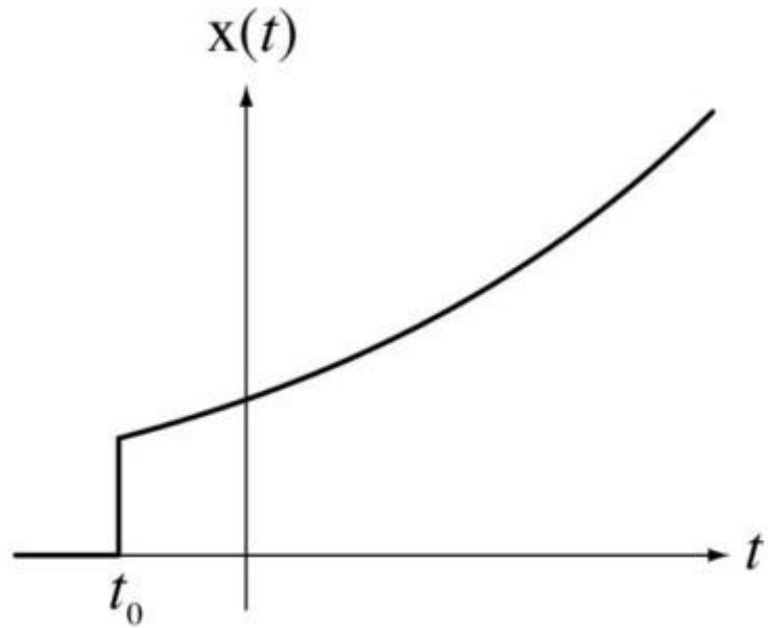
$$x(t) = e^{\alpha t}\,u(t - t_0)\ ,\quad \alpha \in \mathbb{R}$$

$$\int_{t_0}^{\infty} \qquad\qquad \int_{t_0}^{\infty} (\quad)\, t\, e^{-j\omega t}\, dt$$

If $\mathrm{Re}(s) = \sigma > \alpha$ the asymptotic

$(\quad)\, t\, e^{-j\omega t}$ as $t \to \infty$

is to approach zero and the Laplace transform integral converges.



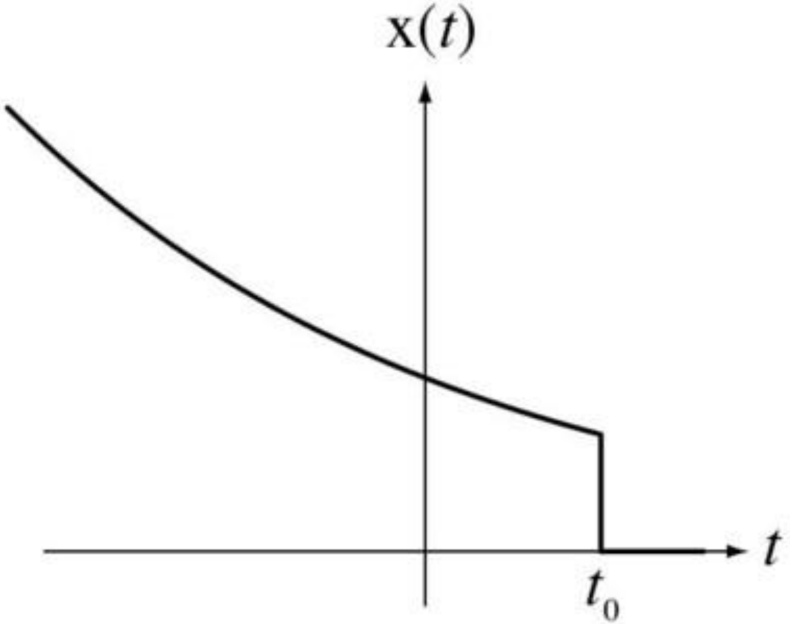
Existence of the Laplace Transform

Left-Sided Exponential

$$x(t) = e^{\beta t}\, u(t_0 - t)\,, \quad \beta \in \mathbb{R}$$

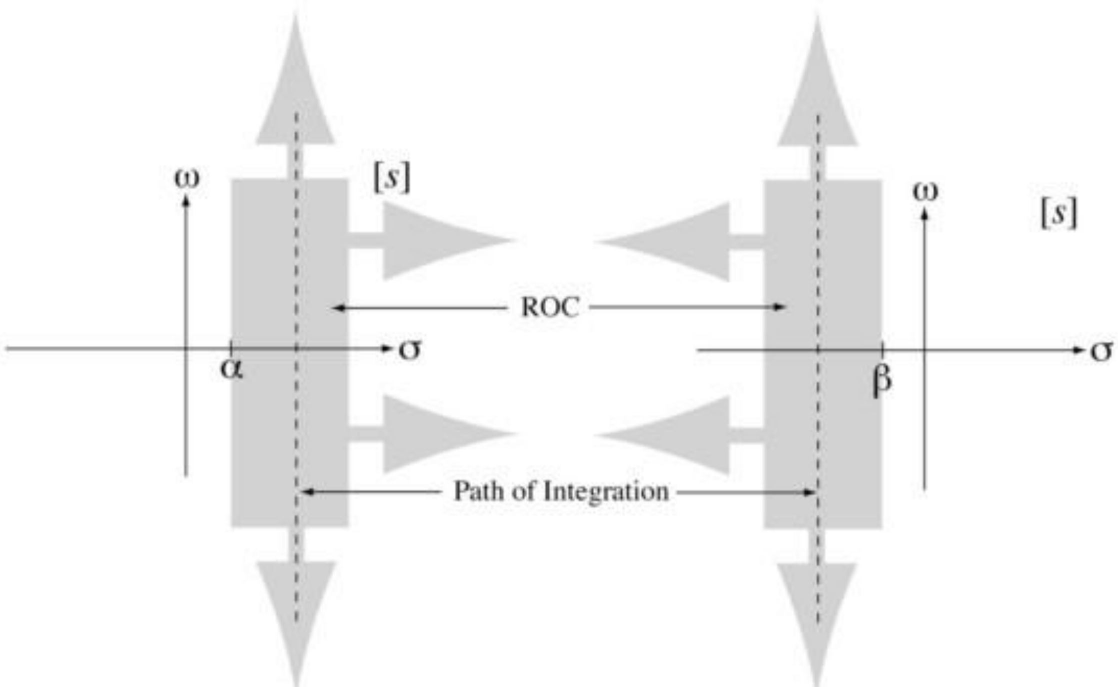
$$X(s) = \int_{-\infty}^{t_0} e^{\beta t} e^{-st}\, dt = \int_{-\infty}^{t_0} e^{(\beta-\sigma)t} e^{-j\omega t}\, dt$$

If $\sigma < \beta$ the asymptotic behavior of $e^{(\beta-\sigma)t}\, e^{-j\omega t}$ as $t \to -\infty$ is to approach zero and the Laplace transform integral converges.



Existence of the Laplace Transform

The two conditions s > a and s < b define the **region of convergence** (**ROC**) for the Laplace transform of right- and left-sided signals.



Existence of the Laplace Transform

Any right-sided signal that grows no faster than an exponential in positive time and any left-sided signal that grows no faster than an exponential in negative time has a Laplace transform.

If $x(t) = x_r(t) + x_l(t)$ where $x_r(t)$ is the right-sided part and $x_l(t)$ is the left-sided part and if $|x_r(t)| < |K_r e^{at}|$ and $|x_l(t)| < |K_l e^{bt}|$ and a and b are as small as possible, then the Laplace-transform integral converges and the Laplace transform exists for $a < s < b$. Therefore if $a < b$ the ROC is the region $a < b$. If $a > b$, there is no ROC and the Laplace transform does not exist.

Laplace Transform Pairs

The Laplace transform of $g_1(t) = Ae^{\alpha t}\,\mathrm{u}(t)$ is

$$G_1(s) = \int_{-\infty}^{\infty} A e^{\alpha t}\,\mathrm{u}(t) e^{-st} dt = A\int_{0}^{\infty} e^{-(s-\alpha)t} dt \quad \int_{0}^{\infty} (\quad)^{t} e^{-j\omega t} dt = \frac{A}{s-\alpha}$$

This function has a **pole** at $s = \alpha$ and the ROC is the region to the right of that point. The Laplace transform of $g_2(t) = Ae^{\beta t}\,\mathrm{u}(-t)$ is

$$G_2(s) = \int_{-\infty}^{\infty} A e^{\beta t}\,\mathrm{u}(-t) e^{-st} dt = A\int_{-\infty}^{0} e^{(\beta-s)t} dt = A\int_{-\infty}^{0} e^{(\beta-s)t} e^{-j\omega t} dt = -\frac{A}{s-\beta}$$

This function has a pole at $s = \beta$ and the ROC is the region to the left of that point.

Region of Convergence

The following two Laplace transform pairs illustrate the importance of the region of convergence.

$$e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha}\ ,\ \sigma > -\alpha$$

$$-e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha}\ ,\ \sigma < -\alpha$$

The two time-domain functions are different but the algebraic expressions for their Laplace transforms are the same. Only the ROC's are different.

Region of Convergence

Some of the most common Laplace transform pairs
(There is more extensive table in the book.)

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1 \ , \ \text{All } \sigma$$

$$\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} 1/s \ , \ \sigma > 0 \qquad -\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} 1/s \ , \ \sigma < 0$$

$$\mathrm{ramp}(t) = t\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} 1/s^2 \ , \ \sigma > 0 \qquad \mathrm{ramp}(-t) = -t\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} 1/s^2 \ , \ \sigma < 0$$

$$e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} 1/(s+\alpha) \ , \ \sigma > -\alpha \qquad -e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} 1/(s+\alpha) \ , \ \sigma < -\alpha$$

$$e^{-\alpha t}\sin(\omega_0 t)\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{(s+\alpha)^2+\omega_0^2} \ , \ \sigma > -\alpha \qquad -e^{-\alpha t}\sin(\omega_0 t)\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{(s+\alpha)^2+\omega_0^2} \ , \ \sigma < -\alpha$$

$$e^{-\alpha t}\cos(\omega_0 t)\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2} \ , \ \sigma > -\alpha \qquad -e^{-\alpha t}\cos(\omega_0 t)\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2} \ , \ \sigma < -\alpha$$

Laplace Transform Example

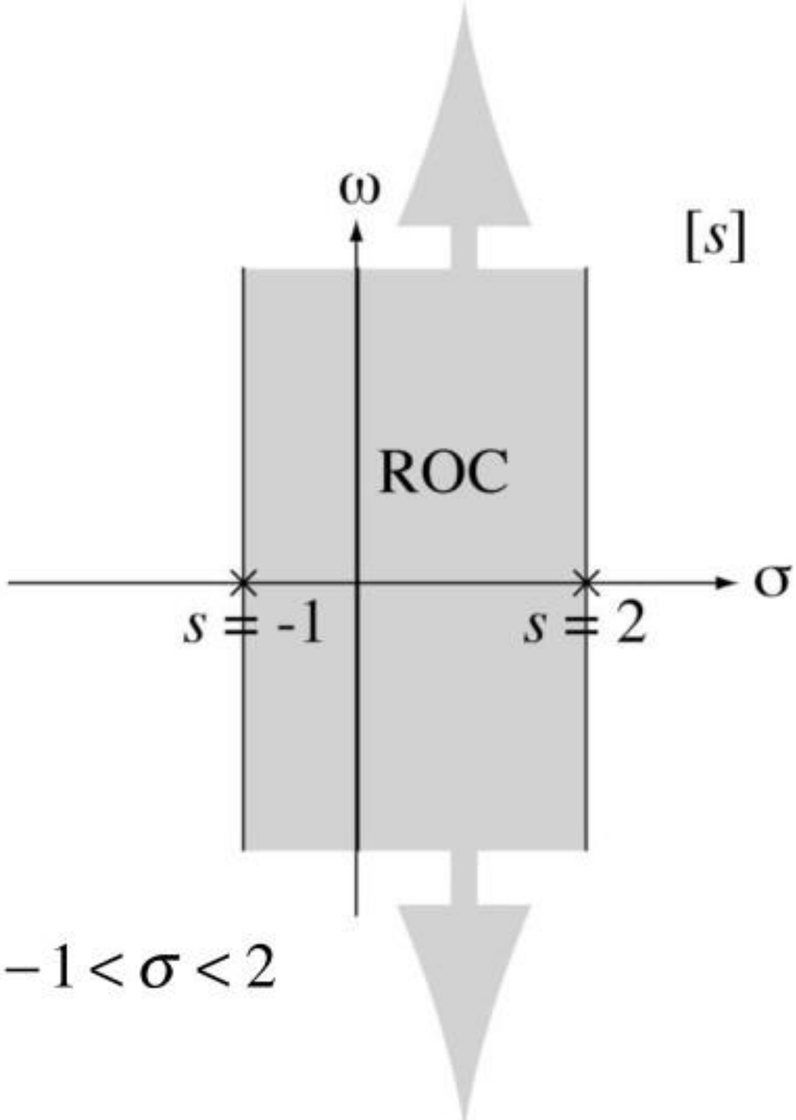
Find the Laplace transform of

$$x(t) = e^{-t}\,u(t) + e^{2t}\,u(-t)$$

$$e^{-t}\,u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} \;,\; \sigma > -1$$

$$e^{2t}\,u(-t) \xleftrightarrow{\mathcal{L}} -\frac{1}{s-2} \;,\; \sigma < 2$$

$$e^{-t}\,u(t) + e^{2t}\,u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} - \frac{1}{s-2} \;,\; -1 < \sigma < 2$$



Laplace Transform Example

Find the inverse Laplace transform of

$$X(s) = \frac{4}{s+3} - \frac{10}{s-6}, \quad -3 < s < 6$$

The ROC tells us that $\frac{4}{s+3}$ must inverse transform into a right-sided signal and that $\frac{10}{s-6}$ must inverse transform into a left-sided signal.

$$x(t) = 4e^{-3t}\,u(t) + 10e^{6t}\,u(-t)$$

Laplace Transform Example

Find the inverse Laplace transform of

$$X(s) = \frac{4}{s+3} - \frac{10}{s-6}, \quad s > 6$$

The ROC tells us that both terms must inverse transform into a right-sided signal.

$$x(t) = 4e^{-3t}u(t) - 10e^{6t}u(t)$$

Laplace Transform Example

Find the inverse Laplace transform of

$$X(s) = \frac{4}{s+3} - \frac{10}{s-6}, \quad s < -3$$

The ROC tells us that both terms must inverse transform into a left-sided signal.

$$x(t) = -4e^{-3t}\,u(-t) + 10e^{6t}\,u(-t)$$

MATLAB System Objects

A MATLAB system object is a special kind of variable in MATLAB that contains all the information about a system. It can be created with the `tf` command whose syntax is

```
sys = tf(num,den)
```

where `num` is a vector of numerator coefficients of powers of s, `den` is a vector of denominator coefficients of powers of s, both in descending order and `sys` is the system object.

MATLAB System Objects

For example, the transfer function

$$H_1(s) = \frac{s^2 + 4}{s^5 + 4s^4 + 7s^3 + 15s^2 + 31s + 75}$$

can be created by the commands

```
»num = [1 0 4] ;  den = [1 4 7 15 31 75] ;

»H1 = tf(num,den) ;

»H1

Transfer function:

                  s^2 + 4

-----------------------------------------

  s^5 + 4 s^4 + 7 s^3 + 15 s^2 + 31 s + 75
```

Partial-Fraction Expansion

The inverse Laplace transform can always be found (in principle at least) by using the inversion integral. But that is rare in engineering practice. The most common type of Laplace-transform expression is a ratio of polynomials in s,

$$G(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \cdots a_1 s + a_0}$$

The denominator can be factored, putting it into the form,

$$G(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{(s - p_1)(s - p_2)\cdots(s - p_N)}$$

Partial-Fraction Expansion

For now, assume that there are no repeated poles and that $N > M$, making the fraction **proper** in s. Then it is possible to write the expression in the **partial fraction** form,

$$G(s) = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_N}{s - p_N}$$

where

$$\frac{b_M s^M + b_{M-1} s^{M-1} + \cdots b_1 s + b_0}{(s - p_1)(s - p_2)\cdots(s - p_N)} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_N}{s - p_N}$$

The K's can be found be any convenient method.

Partial-Fraction Expansion

Multiply both sides by $s - p_1$

$$(s-p_1)\frac{b_M s^M + b_{M-1}s^{M-1} + \cdots + b_1 s + b_0}{(s-p_1)(s-p_2)\cdots(s-p_N)} = \left[\begin{array}{c} K_1 + (s-p_1)\dfrac{K_2}{s-p_2} + \cdots \\ + (s-p_1)\dfrac{K_N}{s-p_N} \end{array}\right]$$

$$K_1 = \frac{b_M p_1^M + b_M p_1^{M-1} + \cdots + b_1 p_1 + b_0}{(p_1-p_2)\cdots(p_1-p_N)}$$

All the K's can be found by the same method and the inverse Laplace transform is then found by table look-up.

Partial-Fraction Expansion

$$H(s) = \frac{10s}{(s+4)(s+9)} = \frac{K_1}{s+4} + \frac{K_2}{s+9}, \quad s > -4$$

$$K_1 = \left[(s+4) \frac{10s}{(s+4)(s+9)} \right]_{s=-4} = \left[\frac{10s}{s+9} \right]_{s=-4} = \frac{-40}{5} = -8$$

$$K_2 = \left[(s+9) \frac{10s}{(s+4)(s+9)} \right]_{s=-9} = \left[\frac{10s}{s+4} \right]_{s=-9} = \frac{-90}{-5} = 18$$

$$H(s) = \frac{-8}{s+4} + \frac{18}{s+9} = \frac{-8s - 72 + 18s + 72}{(s+4)(s+9)} = \frac{10s}{(s+4)(s+9)}$$. Check.

$$h(t) = \left(-8e^{-4t} + 18e^{-9t}\right) u(t)$$

Partial-Fraction Expansion

If the expression has a repeated pole of the form,

$$G(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{(s - p_1)^2 (s - p_3) \cdots (s - p_N)}$$

the partial fraction expansion is of the form,

$$G(s) = \frac{K_{12}}{(s - p_1)^2} + \frac{K_{11}}{s - p_1} + \frac{K_3}{s - p_3} + \cdots + \frac{K_N}{s - p_N}$$

and K_{12} can be found using the same method as before.

But K_{11} cannot be found using the same method.

Partial-Fraction Expansion

Instead K_{11} can be found by using the more general formula

$$K_{qk} = \frac{1}{(m-k)!} \frac{d^{m-k}}{ds^{m-k}} \left[\left(s - p_q \right)^m \mathrm{H}(s) \right]_{s \to p_q} , \quad k = 1, 2, \cdots, m$$

where m is the order of the qth pole, which applies to repeated poles of any order.

If the expression is not a proper fraction in s the partial-fraction method will not work. But it is always possible to **synthetically divide** the numerator by the denominator until the remainder is a proper fraction and then apply partial-fraction expansion.

Partial-Fraction Expansion

$$H(s) = \frac{10s}{(s+4)^2(s+9)} = \frac{K_{12}}{(s+4)^2} + \frac{K_{11}}{s+4} + \frac{K_2}{s+9}, \quad s > 4$$

-

Repeated Pole

$$K_{12} = \left[(s+4)^2 \frac{10s}{(s+4)^2(s+9)} \right]_{s=-4} = \frac{-40}{5} = -8$$

Using

$$K_{qk} = \frac{1}{(m-k)!} \frac{d^{m-k}}{ds^{m-k}} \left[(s-p_q)^m H(s) \right]_{s \to p_q}, \quad k = 1, 2, \cdots, m$$

$$K_{11} = \frac{1}{(2-1)!} \frac{d^{2-1}}{ds^{2-1}} \left[(s+4)^2 H(s) \right]_{s \to -4} = \frac{d}{ds} \left[\frac{10s}{\;} \right]_{s \to -4}$$

Partial-Fraction Expansion

$$K_1 = \left[\frac{(s+9)10 - 10s}{(s+9)^2}\right]_{s=-4} = \frac{18}{5}$$

$$K_2 = -\frac{18}{5} \Rightarrow H(s) = \frac{-8}{(s+4)^2} + \frac{8/5}{s+4} + \frac{-18/5}{s+9}, \quad s > -4$$

$$H(s) = \frac{-8s - 72 + \frac{18}{5}\left(s^2 + 13s + 36\right) - \frac{18}{5}\left(s^2 + 8s + 16\right)}{(s+4)^2(s+9)}, \quad s > -4$$

$$H(s) = \frac{10s}{(s+4)^2(s+9)}, \quad s > -4$$

$$h(t) = \left(-8te^{-4t} + \frac{18}{5}e^{-4t} - \frac{18}{5}e^{-9t}\right)u(t)$$

Partial-Fraction Expansion

$$\mathrm{H}(s)=\frac{10s^2}{(s+4)(s+9)} \;,\; s > -4 \leftarrow \textbf{Improper in s}$$

$$\mathrm{H}(s)=\frac{10s^2}{s^2+13s+36} \;,\; s > -4$$

Synthetic Division → $s^2 + 13s + 36 \,\overline{\big)\, 10s^2}$ with quotient 10

$$\begin{array}{r} 10 \\ s^2+13s+36\;\big)\;10s^2 \qquad\qquad\quad \\ \underline{10s^2+130s+360} \\ -130s-360 \end{array}$$

$$\mathrm{H}(s)=10-\frac{130s+360}{(s+4)(s+9)}=10-\left[\frac{-32}{s+4}+\frac{162}{s+9}\right] \;,\; s > -4$$

$$\mathrm{h}(t)=10\delta(t)-\left[162e^{-9t}-32e^{-4t}\right]\mathrm{u}(t)$$

Inverse Laplace Transform Example

Method 1

$$G(s) = \frac{s}{(s-3)(s^2-4s+5)} \, , \quad \sigma < 2$$

$$G(s) = \frac{s}{(s-3)(s-2+j)(s-2-j)} \, , \quad \sigma < 2$$

$$G(s) = \frac{3/2}{s-3} - \frac{(3+j)/4}{s-2+j} - \frac{(3-j)/4}{s-2-j} \, , \quad \sigma < 2$$

$$g(t) = \left(-\frac{3}{2} e^{3t} + \frac{3+j}{4} e^{(2-j)t} + \frac{3-j}{4} e^{(2+j)t} \right) u(-t)$$

Inverse Laplace Transform Example

$$g(t) = \left(-\frac{3}{2}e^{3t} + \frac{3+j}{4}e^{(2-j)t} + \frac{3-j}{4}e^{(2+j)t} \right) u(-t)$$

This looks like a function of time that is complex-valued. But, with the use of some trigonometric identities it can be put into the form

$$g(t) = (3/2)\left\{ e^{2t}\left[\cos(t) + (1/3)\sin(t)\right] - e^{3t} \right\} u(-t)$$

which has only real values.

Inverse Laplace Transform Example

Method 2

$$G(s) = \frac{s}{(s-3)(s^2-4s+5)} \quad , \quad s < 2$$

$$G(s) = \frac{s}{(s-3)(s-2+j)(s-2-j)} \quad , \quad s < 2$$

$$G(s) = \frac{3/2}{s-3} - \frac{(3+j)/4}{s-2+j} - \frac{(3-j)/4}{s-2-j}, \quad s < 2$$

Getting a common denominator and simplifying

$$G(s) = \frac{3/2}{s-3} - \frac{1}{4}\frac{6s-10}{s^2-4s+5} = \frac{3/2}{s-3} - \frac{6}{4}\frac{s-5/3}{(s-2)^2+1} \quad , \quad s < 2$$

Inverse Laplace Transform Example

Method 2

$$G(s)=\frac{3/2}{s-3}-\frac{6}{4}\frac{s-5/3}{(s-2)^2+1}\ ,\quad s<2$$

The denominator of the second term has the form of the Laplace transform of a damped cosine or damped sine but the numerator is not yet in the correct form. But by adding and subtracting the correct expression from that term and factoring we can put it into the form

$$G(s)=\frac{3/2}{s-3}-\frac{3}{2}\left[\frac{s-2}{(s-2)^2+1}+\frac{1/3}{(s-2)^2+1}\right],\quad s<2$$

Inverse Laplace Transform Example

Method 2

$$G(s) = \frac{3/2}{s-3} - \frac{3}{2}\left[\frac{s-2}{(s-2)^2+1} + \frac{1/3}{(s-2)^2+1}\right], \quad s < 2$$

This can now be directly inverse Laplace transformed into

$$g(t) = (3/2)\left\{e^{2t}\left[\cos(t) + (1/3)\sin(t)\right] - e^{3t}\right\}u(-t)$$

which is the same as the previous result.

Inverse Laplace Transform Example

Method 3

When we have a pair of poles p_2 and p_3 that are complex conjugates

we can convert the form $G(s) = \frac{A}{s-3} + \frac{K_2}{s-p_2} + \frac{K_3}{s-p_3}$ into the

form $G(s) = \frac{A}{s-3} + \frac{s(K_2 + K_3) - K_3 p_2 - K_2 p_3}{s^2 - (p_1 + p_2)s + p_1 p_2} = \frac{A}{s-3} + \frac{Bs + C}{s^2 - (p_1 + p_2)s + p_1 p_2}$

In this example we can find the constants A, B and C by realizing that

$$G(s) = \frac{s}{(s-3)(s^2 - 4s + 5)} \quad \frac{\quad}{s-3} + \frac{Bs+C}{s^2 - 4s + 5}, \quad s < 2$$

is not just an equation, it is an **identity**. That means it must be an equality for any value of s.

Inverse Laplace Transform Example

Method 3

A can be found as before to be 3/ 2. Letting $s = 0$, the identity becomes $0 \equiv \frac{-3/2}{3} + \frac{C}{5}$ and $C = 5/2$. Then, letting $s = 1$, and solving we get $B = -3/2$. Now

$$G(s) = \frac{3/2}{s-3} + \frac{(-3/2)s + 5/2}{s^2 - 4s + 5} \quad , \quad s < 2$$

or

$$G(s) = \frac{3/2}{s-3} - \frac{3}{2}\frac{s - 5/3}{s^2 - 4s + 5} \quad , \quad s < 2$$

This is the same as a result in Method 2 and the rest of the solution is also the same. The advantage of this method is that all the numbers are real.

Use of MATLAB in Partial Fraction Expansion

MATLAB has a function `residue` that can be very helpful in partial fraction expansion. Its syntax is `[r,p,k] = residue(b,a)` where `b` is a vector of coefficients of descending powers of s in the numerator of the expression and `a` is a vector of coefficients of descending powers of s in the denominator of the expression, `r` is a vector of residues, `p` is a vector of finite pole locations and `k` is a vector of so-called direct terms which result when the degree of the numerator is equal to or greater than the degree of the denominator. For our purposes, residues are simply the numerators in the partial-fraction expansion.

Laplace Transform Properties

Let $g(t)$ and $h(t)$ form the transform pairs, $g(t) \xleftrightarrow{\mathcal{L}} G(s)$ and $h(t) \xleftrightarrow{\mathcal{L}} H(s)$ with ROC's, ROC_G and ROC_H respectively.

Linearity $\alpha g(t) + \beta h(t) \xleftrightarrow{\mathcal{L}} \alpha G(s) + \beta H(s)$

$$ROC \supseteq ROC_G \cap ROC_H$$

Time Shifting $g(t - t_0) \xleftrightarrow{\mathcal{L}} G(s) e^{-st_0}$

$$ROC = ROC_G$$

***s*-Domain Shift** $e^{s_0 t} g(t) \xleftrightarrow{\mathcal{L}} G(s - s_0)$

$ROC = ROC_G$ shifted by s_0,

(s is in ROC if $s - s_0$ is in ROC_G)

Laplace Transform Properties

Time Scaling

$$g(at) \xleftrightarrow{\mathcal{L}} (1/|a|)G(s/a)$$

$\text{ROC} = \text{ROC}_G$ scaled by a

(s is in ROC if s/a is in ROC_G)

Time Differentiation

$$\frac{d}{dt}g(t) \xleftrightarrow{\mathcal{L}} sG(s)$$

$\text{ROC} \supseteq \text{ROC}_G$

***s*-Domain Differentiation**

$$-t\,g(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}G(s)$$

$\text{ROC} = \text{ROC}_G$

Laplace Transform Properties

Convolution in Time $\quad \mathrm{g}(t) * \mathrm{h}(t) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)\mathrm{H}(s)$

$$\mathrm{ROC} \supseteq \mathrm{ROC}_{\mathrm{G}} \cap \mathrm{ROC}_{\mathrm{H}}$$

Time Integration $\quad \int_{-\infty}^{t} \mathrm{g}(\tau)\,d\tau \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)/s$

$$\mathrm{ROC} \supseteq \mathrm{ROC}_{\mathrm{G}} \cap (\sigma > 0)$$

If $\mathrm{g}(t) = 0$, $t < 0$ and there are no impulses or higher-order singularities at $t = 0$ then

Initial Value Theorem: $\quad \mathrm{g}(0^{+}) = \lim_{s \to \infty} s\,\mathrm{G}(s)$

Final Value Theorem: $\quad \lim_{t \to \infty} \mathrm{g}(t) = \lim_{s \to 0} s\,\mathrm{G}(s)$ if $\lim_{t \to \infty} \mathrm{g}(t)$ exists

Laplace Transform Properties

Final Value Theorem $\lim_{t\to\infty} g(t) = \lim_{s\to 0} s\,G(s)$

This theorem only applies if the limit $\lim_{t\to\infty} g(t)$ actually exists.

It is possible for the limit $\lim_{s\to 0} s\,G(s)$ to exist even though the limit $\lim_{t\to\infty} g(t)$ does not exist. For example

$$x(t) = \cos(\omega_0 t) \xleftrightarrow{\mathcal{L}} X(s) = \frac{s}{s^2 + \omega_0^2}$$

$$\lim_{s\to 0} s\,X(s) = \lim_{s\to 0} \frac{s^2}{s^2 + \omega_0^2} = 0$$

but $\lim_{t\to\infty} \cos(\omega_0 t)$ does not exist.

Laplace Transform Properties

Final Value Theorem

The final value theorem applies to a function $\mathrm{G}(s)$ if all the poles of $s\mathrm{G}(s)$ lie in the open left half of the s plane. Be sure to notice that this does not say that all the poles of $\mathrm{G}(s)$ must lie in the open left half of the s plane. $\mathrm{G}(s)$ could have a single pole at $s = 0$ and the final value theorem would still apply.

Use of Laplace Transform Properties

Find the Laplace transforms of $\mathrm{x}(t) = \mathrm{u}(t) - \mathrm{u}(t-a)$ and $\mathrm{x}(2t) = \mathrm{u}(2t) - \mathrm{u}(2t-a)$. From the table $\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} 1/s, \sigma > 0.$

Then, using the time-shifting property $\mathrm{u}(t-a) \xleftrightarrow{\mathcal{L}} e^{-as}/s, \sigma > 0.$

Using the linearity property $\mathrm{u}(t) - \mathrm{u}(t-a) \xleftrightarrow{\mathcal{L}} \left(1 - e^{-as}\right)/s, \sigma > 0.$

Using the time-scaling property

$$\mathrm{u}(2t) - \mathrm{u}(2t-a) \xleftrightarrow{\mathcal{L}} \frac{1}{2}\left[\frac{1-e^{-as}}{s}\right]_{s \to s/2} = \frac{1-e^{-as/2}}{s}, \sigma > 0$$

Use of Laplace Transform Properties

Use the s-domain differentiation property and

$$\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} 1/s \ , \sigma > 0$$

to find the inverse Laplace transform of $1/s^2$. The s-domain differentiation property is $-t\,\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}(\mathrm{G}(s))$. Then $-t\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}\left(\frac{1}{s}\right) = -\frac{1}{s^2}$. Then using the linearity property

$$t\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}.$$

The Unilateral Laplace Transform

In most practical signal and system analysis using the Laplace transform a modified form of the transform, called the **unilateral Laplace transform**, is used. The unilateral Laplace transform is defined by $G(s) = \int_{0^-}^{\infty} g(t)\, e^{-st}\, dt$. The only difference between this version and the previous definition is the change of the lower integration limit from $-\infty$ to 0^-. With this definition, all the Laplace transforms of causal functions are the same as before with the same ROC, the region of the s plane to the right of all the finite poles.

The Unilateral Laplace Transform

The unilateral Laplace transform integral excludes negative time. If a function has non-zero behavior in negative time its unilateral and bilateral transforms will be different. Also functions with the same positive time behavior but different negative time behavior will have the same unilateral Laplace transform. Therefore, to avoid ambiguity and confusion, the unilateral Laplace transform should only be used in analysis of causal signals and systems. This is a limitation but in most practical analysis this limitation is not significant and the unilateral Laplace transform actually has advantages.

The Unilateral Laplace Transform

The main advantage of the unilateral Laplace transform is that the ROC is simpler than for the bilateral Laplace transform and, in most practical analysis, involved consideration of the ROC is unnecessary. The inverse Laplace transform is unchanged. It is

$$g(t) = \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} G(s) e^{+st} ds$$

The Unilateral Laplace Transform

Some of the properties of the unilateral Laplace transform are different from the bilateral Laplace transform.

Time-Shifting $\quad \mathrm{g}(t-t_0) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)e^{-st_0} \;, t_0 > 0$

Time Scaling $\quad \mathrm{g}(at) \xleftrightarrow{\mathcal{L}} (1/|a|)\mathrm{G}(s/a) \;, a > 0$

First Time Derivative $\quad \frac{d}{dt}\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} s\mathrm{G}(s) - \mathrm{g}(0^-)$

Nth Time Derivative $\quad \frac{d^N}{dt^N}(\mathrm{g}(t)) \xleftrightarrow{\mathcal{L}} s^N \mathrm{G}(s) - \sum_{n=1}^{N} s^{N-n}\left[\frac{d^{n-1}}{dt^{n-1}}(\mathrm{g}(t))\right]_{t=0^-}$

Time Integration $\quad \int_{0^-}^{t} \mathrm{g}(\tau)d\tau \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)/s$

The Unilateral Laplace Transform

The time shifting property applies only for shifts to the right because a shift to the left could cause a signal to become non-causal. For the same reason scaling in time must only be done with positive scaling coefficients so that time is not reversed producing an anti-causal function. The derivative property must now take into account the initial value of the function at time $t = 0^-$ and the integral property applies only to functional behavior after time $t = 0$. Since the unilateral and bilateral Laplace transforms are the same for causal functions, the bilateral table of transform pairs can be used for causal functions.

The Unilateral Laplace Transform

The Laplace transform was developed for the solution of differential equations and the unilateral form is especially well suited for solving differential equations with initial conditions. For example,

$$\frac{d^2}{dt^2}\left[x(t)\right] + 7\frac{d}{dt}\left[x(t)\right] + 12x(t) = 0$$

with initial conditions $x(0^-) = 2$ and $\frac{d}{dt}(x(t))_{t=0^-} = -4.$

Laplace transforming both sides of the equation, using the new derivative property for unilateral Laplace transforms,

$$s^2 X(s) - s x(0^-) - \frac{d}{dt}(x(t))_{t=0^-} + 7\left[\qquad \right] + 12X(s) = 0$$

The Unilateral Laplace Transform

Solving for $X(s)$

$$X(s) = \frac{s\overbrace{x(0^-)}^{=2} + 7\overbrace{x(0^-)}^{=2} + \overbrace{\frac{d}{dt}(x(t))_{t=0^-}}^{=-4}}{s^2 + 7s + 12}$$

or $X(s) = \dfrac{2s+10}{s^2+7s+12} = \dfrac{4}{s+3} - \dfrac{2}{s+4}$. The inverse transform yields

$x(t) = \left(4e^{-3t} - 2e^{-4t}\right)u(t)$. This solution solves the differential equation with the given initial conditions.

Pole-Zero Diagrams and Frequency Response

If the transfer function of a stable system is $\text{H}(s)$, the frequency response is $\text{H}(j\omega)$. The most common type of transfer function is of the form,

$$\text{H}(s) = A\frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}$$

Therefore $\text{H}(j\omega)$ is

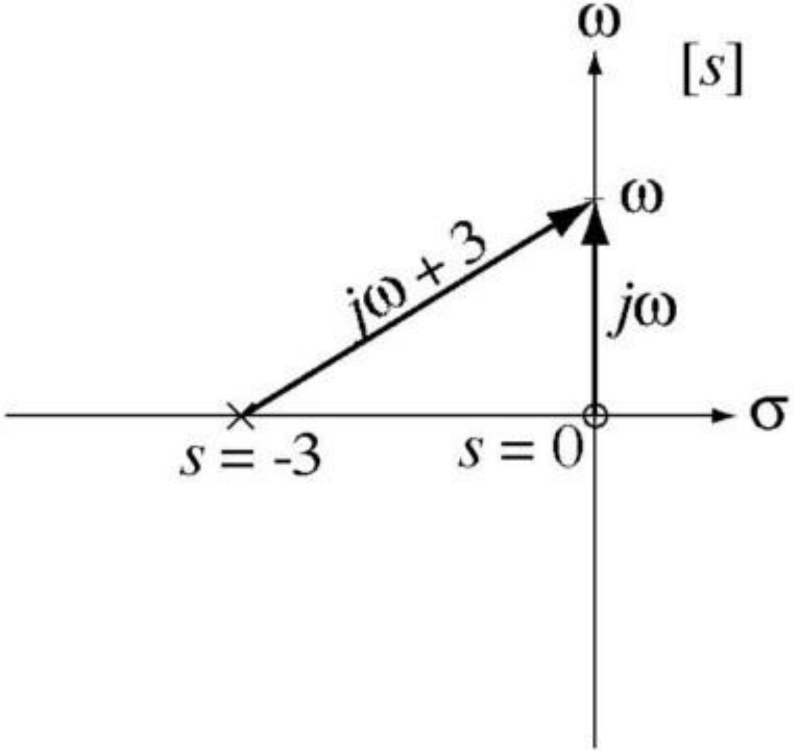
$$\text{H}(j\omega) = A\frac{(j\omega-z_1)(j\omega-z_2)\cdots(j\omega-z_M)}{(j\omega-p_1)(j\omega-p_2)\cdots(j\omega-p_N)}$$

Pole-Zero Diagrams and Frequency Response

Let $H(s) = \frac{3s}{s+3}$.

$$H(j\omega) = 3\frac{j\omega}{j\omega + 3}$$

The numerator $j\omega$ and the denominator $j\omega + 3$ can be conceived as vectors in the s plane.



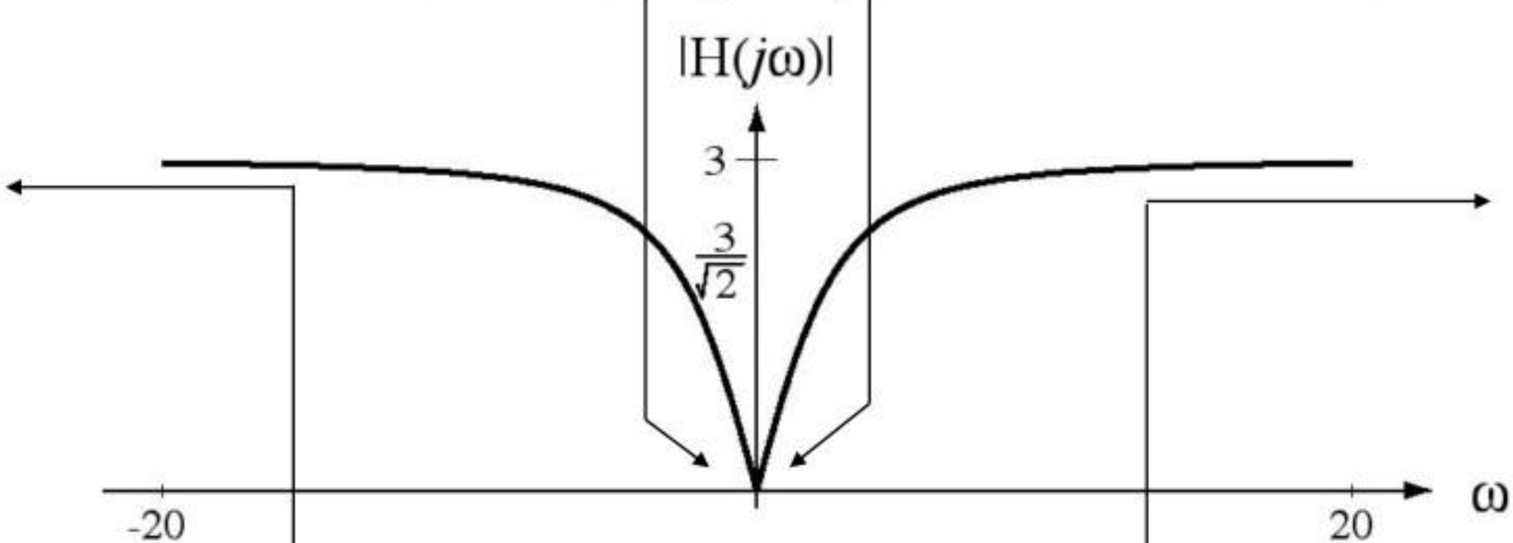
$$|H(j\omega)| = 3\frac{|j\omega|}{|j\omega + 3|}$$

$$\measuredangle H(j\omega) = \underbrace{\measuredangle 3}_{=0} + \measuredangle j\omega - \measuredangle(j\omega + 3)$$

Pole-Zero Diagrams and Frequency Response

$$\lim_{\omega \to 0^-} |H(j\omega)| = \lim_{\omega \to 0^-} 3 \frac{|j\omega|}{|j\omega + 3|} = 0$$

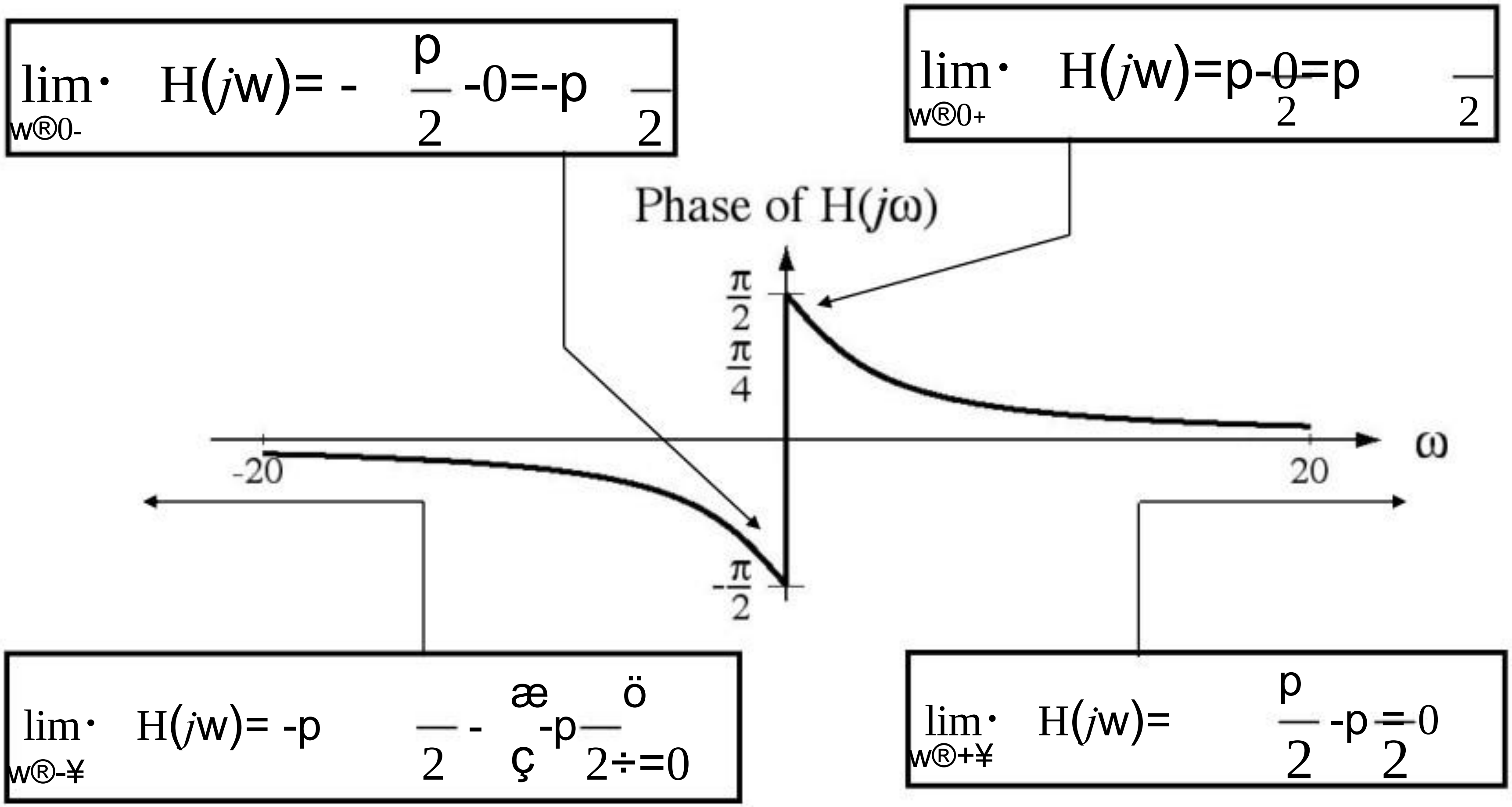
$$\lim_{\omega \to 0^+} |H(j\omega)| = \lim_{\omega \to 0^+} 3 \frac{|j\omega|}{|j\omega + 3|} = 0$$



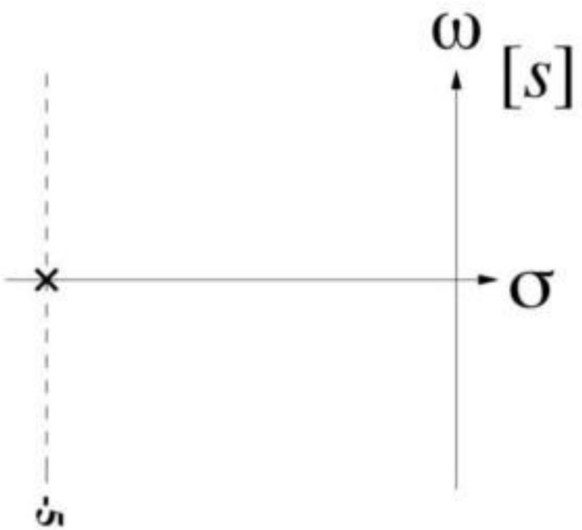
$$\lim_{\omega \to -\infty} |H(j\omega)| = \lim_{\omega \to -\infty} 3 \frac{|j\omega|}{|j\omega + 3|} = 3$$

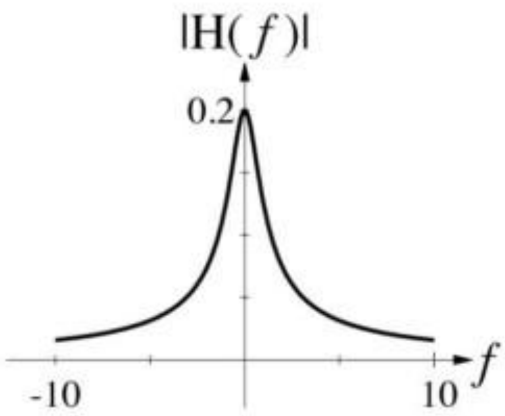
$$\lim_{\omega \to +\infty} |H(j\omega)| = \lim_{\omega \to +\infty} 3 \frac{|j\omega|}{|j\omega + 3|} = 3$$

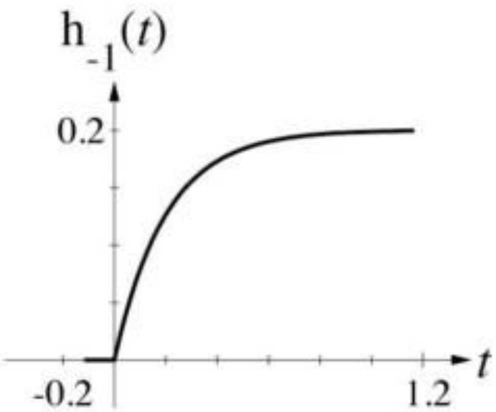
Pole-Zero Diagrams and Frequency Response



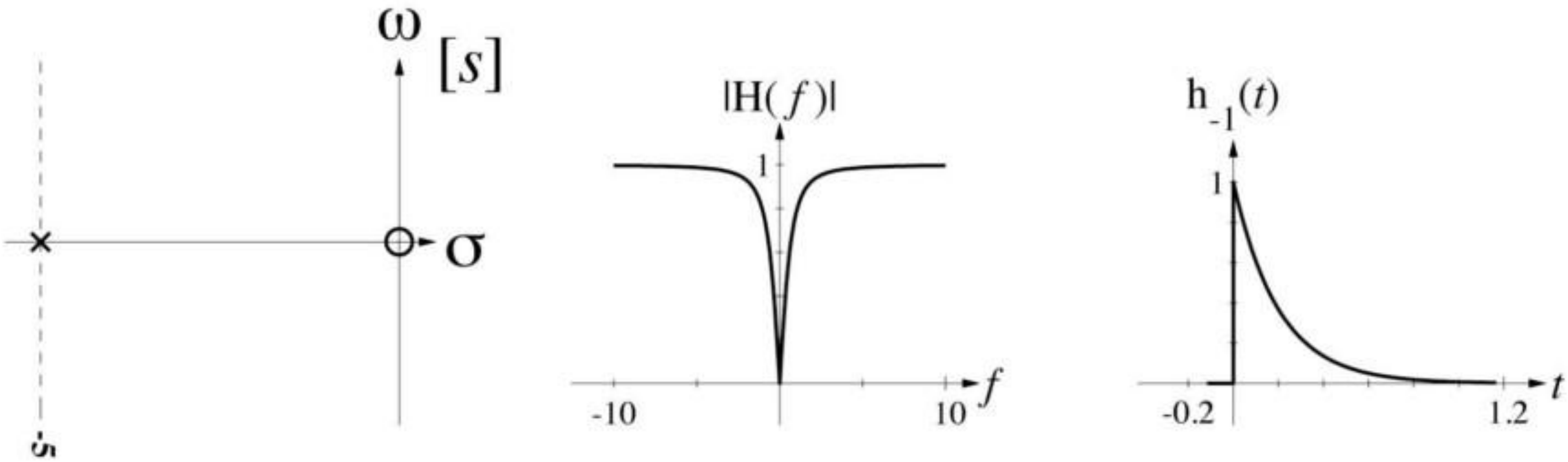
Pole-Zero Diagrams and Frequency Response



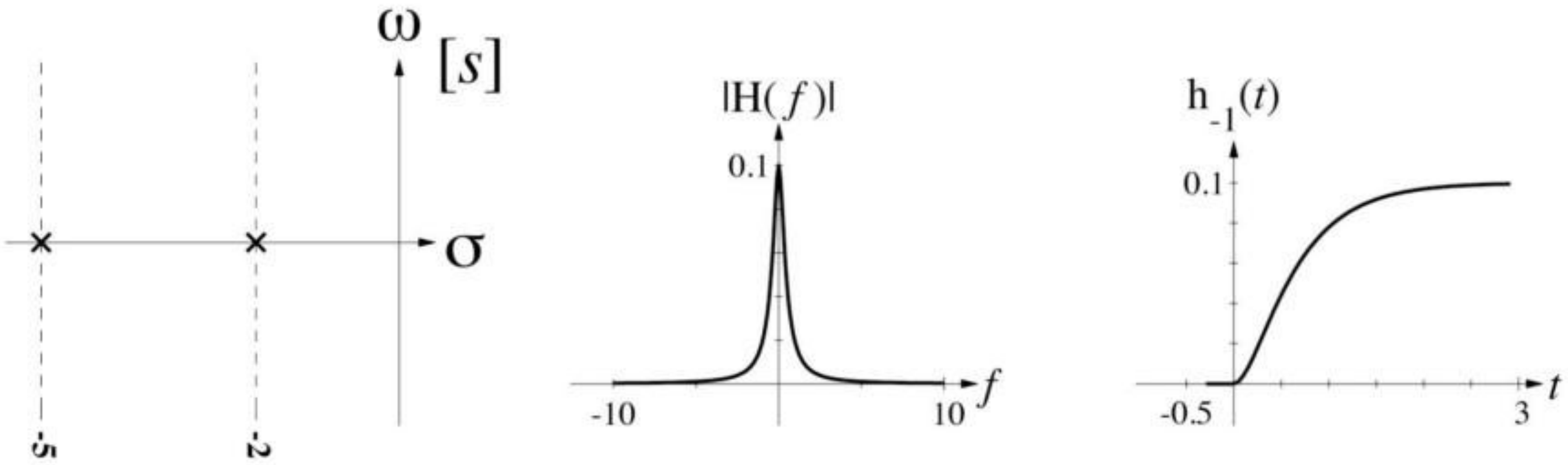




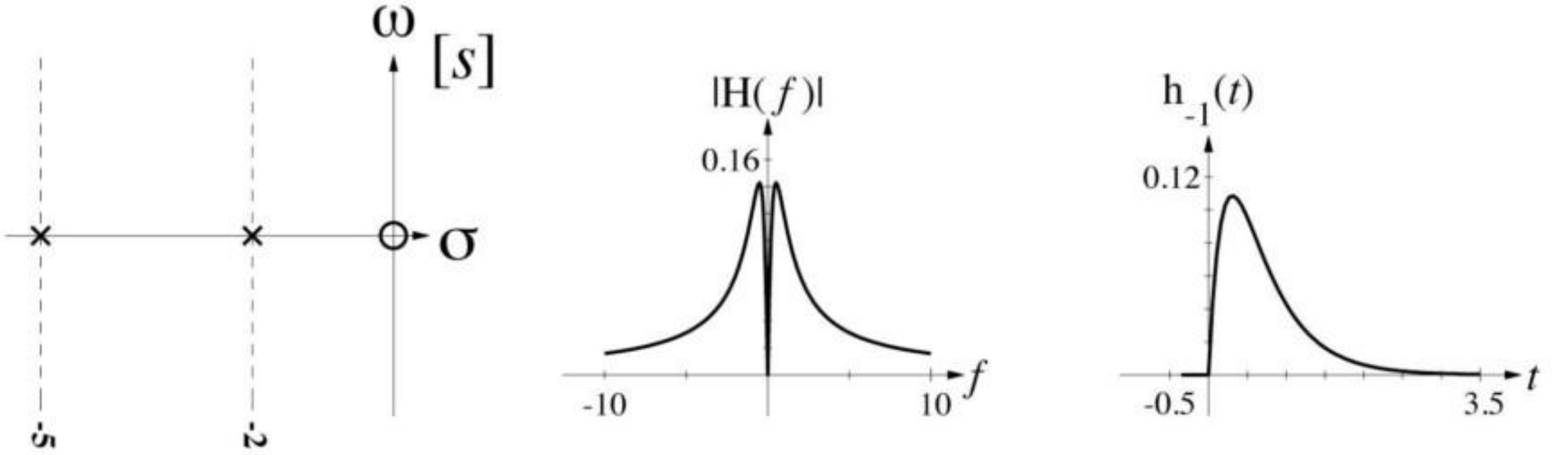
Pole-Zero Diagrams and Frequency Response



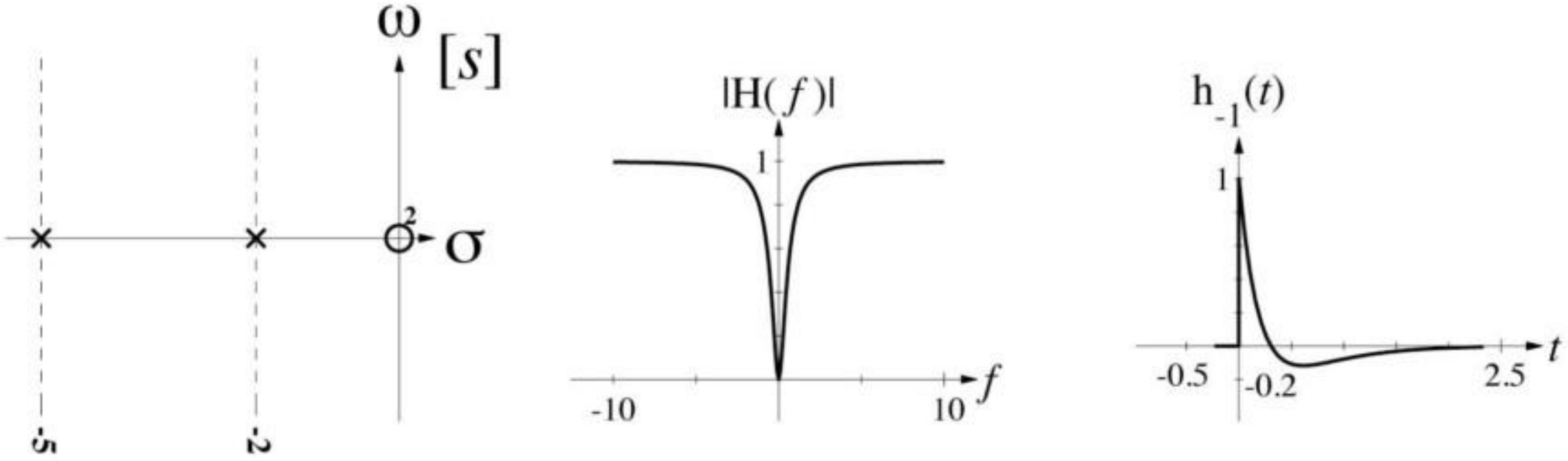
Pole-Zero Diagrams and Frequency Response



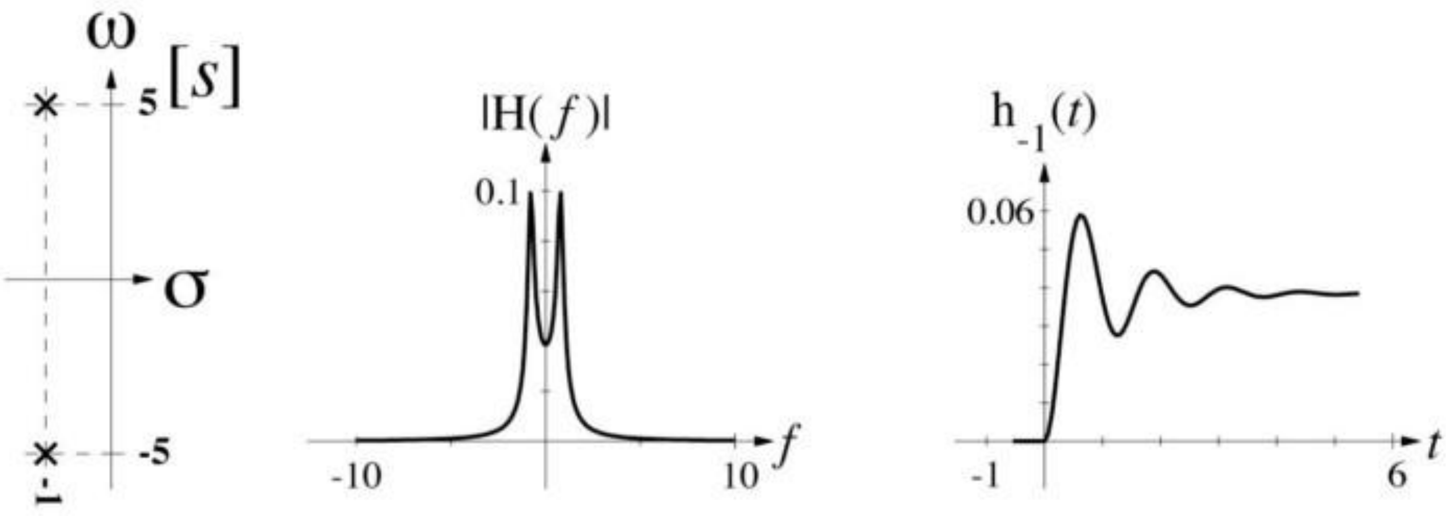
Pole-Zero Diagrams and Frequency Response



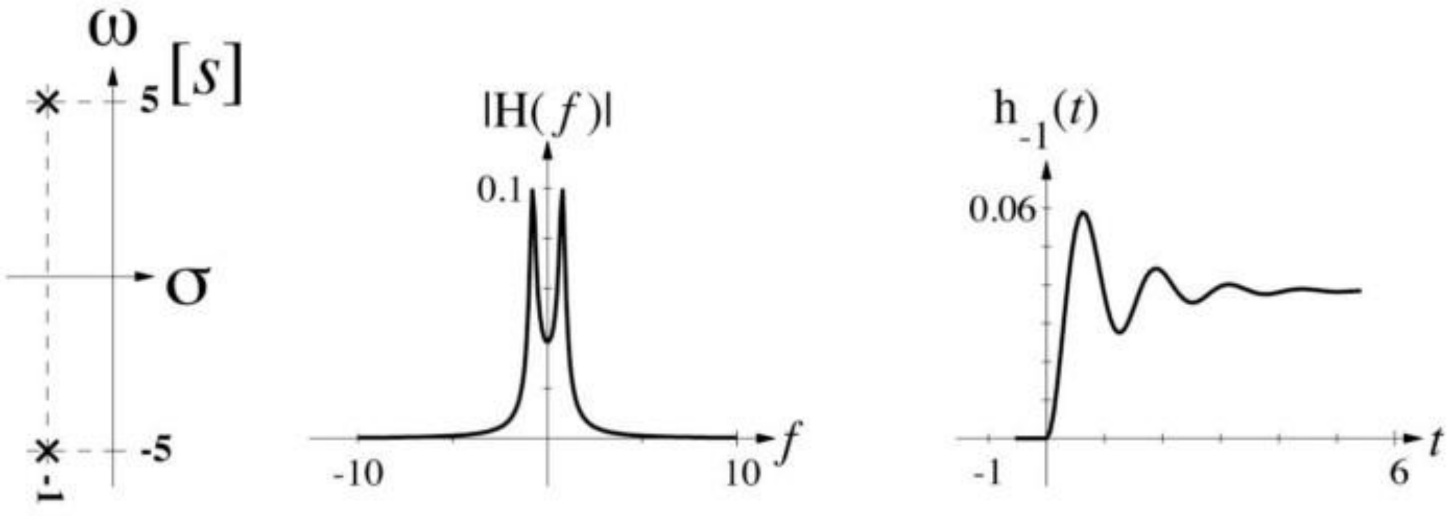
Pole-Zero Diagrams and Frequency Response



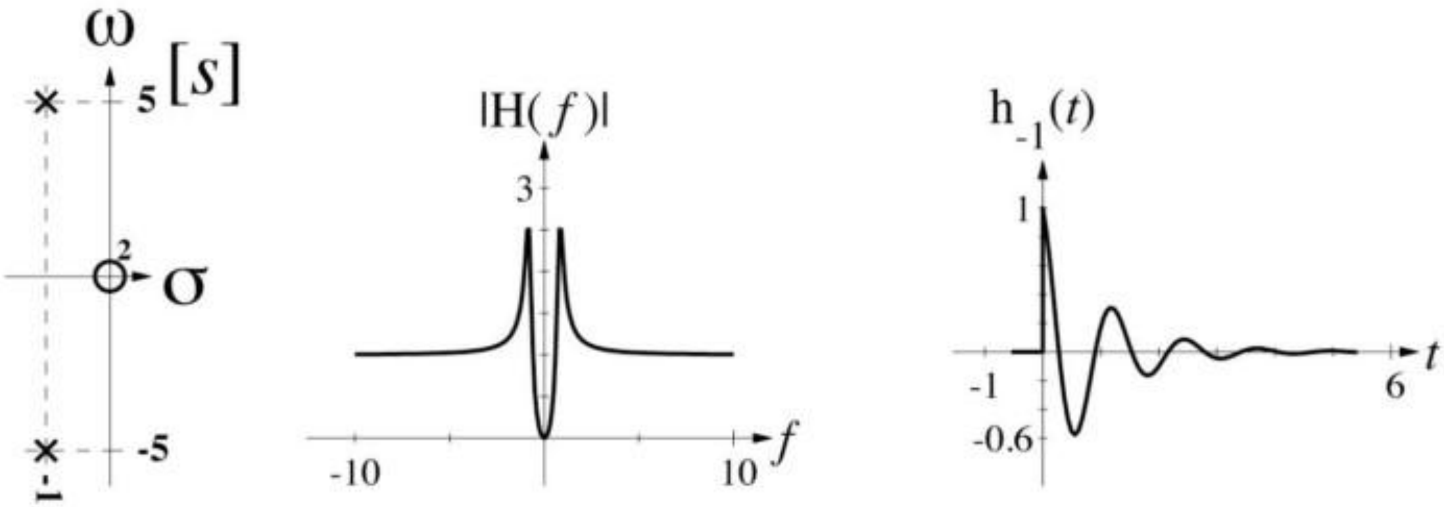
Pole-Zero Diagrams and Frequency Response



Pole-Zero Diagrams and Frequency Response



Pole-Zero Diagrams and Frequency Response



The *z* Transform

Generalizing the DTFT

The forward DTFT is defined by $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$ in which

Ω is discrete-time radian frequency, a real variable. The quantity $e^{j\Omega n}$ is then a complex sinusoid whose magnitude is always one and whose phase can range over all angles. It always lies on the unit circle in the complex plane. If we now replace $e^{j\Omega}$ with a variable z that can have any complex value we define the z transform $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

The DTFT expresses signals as linear combinations of complex sinusoids. The z transform expresses signals as linear combinations of complex exponentials.

Complex Exponential Excitation

Let the excitation of a discrete-time LTI system be a complex exponential of the form Az^n where z is, in general, complex and A is any constant. Using convolution, the response $\mathrm{y}[n]$ of an LTI system with impulse response $\mathrm{h}[n]$ to a complex exponential excitation $\mathrm{x}[n]$ is

$$\mathrm{y}[n] = \mathrm{h}[n] * Az^n = A \sum_{m=-\infty}^{\infty} \mathrm{h}[m] z^{n-m} = \underbrace{Az^n}_{=\mathrm{x}[n]} \sum_{m=-\infty}^{\infty} \mathrm{h}[m] z^{-m}$$

The response is the product of the excitation and the z transform of $\mathrm{h}[n]$ defined by $\mathrm{H}(z) = \sum_{m=-\infty}^{\infty} \mathrm{h}[n] z^{-n}$.

The Transfer Function

If an LTI system with impulse response $\mathrm{h}[n]$ is excited by a signal, $\mathrm{x}[n]$, the z transform $\mathrm{Y}(z)$ of the response $\mathrm{y}[n]$ is

$$\mathrm{Y}(z)=\sum_{n=-\infty}^{\infty}\mathrm{y}[n]z^{-n}=\sum_{n=-\infty}^{\infty}(\mathrm{h}[n]*\mathrm{x}[n])z^{-n}=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\mathrm{h}[m]\mathrm{x}[n-m]z^{-n}$$

$$\mathrm{Y}(z)=\sum_{m=-\infty}^{\infty}\mathrm{h}[m]\sum_{n=-\infty}^{\infty}\mathrm{x}[n-m]z^{-n}$$

Let $q=n-m$. Then

$$\mathrm{Y}(z)=\sum_{m=-\infty}^{\infty}\mathrm{h}[m]\sum_{q=-\infty}^{\infty}\mathrm{x}[q]z^{-(q+m)}=\underbrace{\sum_{m=-\infty}^{\infty}\mathrm{h}[m]z^{-m}}_{=\mathrm{H}(z)}\underbrace{\sum_{q=-\infty}^{\infty}\mathrm{x}[q]z^{-q}}_{=\mathrm{X}(z)}$$

$$\mathrm{Y}(z)=\mathrm{H}(z)\mathrm{X}(z)$$

$\mathrm{H}(z)$ is the **transfer function**.

Systems Described by Difference Equations

The most common description of a discrete-time system is a difference equation of the general form

$$\sum_{k=0}^{N} a_k \, \mathrm{y}[n-k] = \sum_{k=0}^{M} b_k \, \mathrm{x}[n-k].$$

It was shown in Chapter 5 that the transfer function for a system of this type is

$$\mathrm{H}(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

or

$$\mathrm{H}(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = z^{N-M} \frac{b_0 z^{M} + b_1 z^{M-1} + \cdots + b_{M-1} z + b_M}{a_0 z^{N} + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}$$

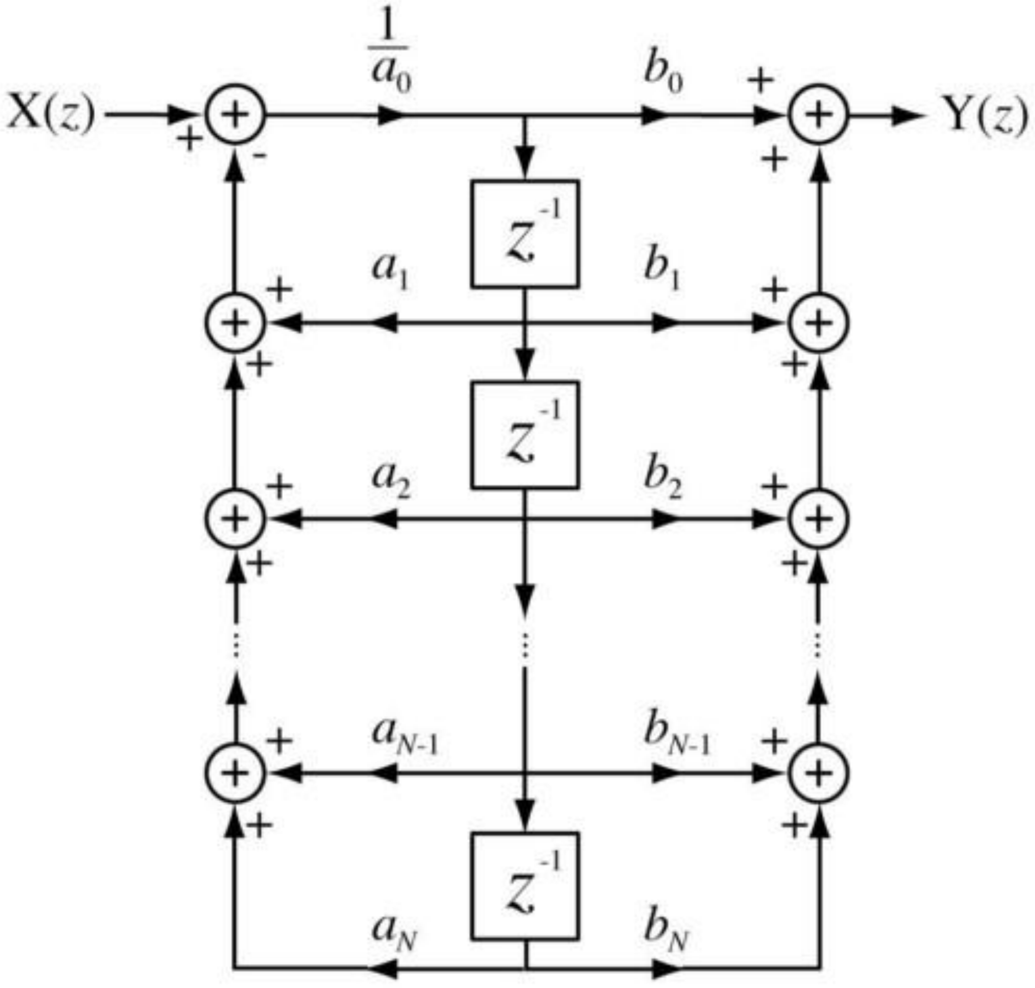
Direct Form II Realization

Direct Form II realization of a discrete-time system is similar in form to Direct Form II realization of continuous-time systems

A continuous-time system can be realized with integrators, summing junctions and multipliers

A discrete-time system can be realized with delays, summing junctions and multipliers

Direct Form II Realization



The Inverse z Transform

The inversion integral is

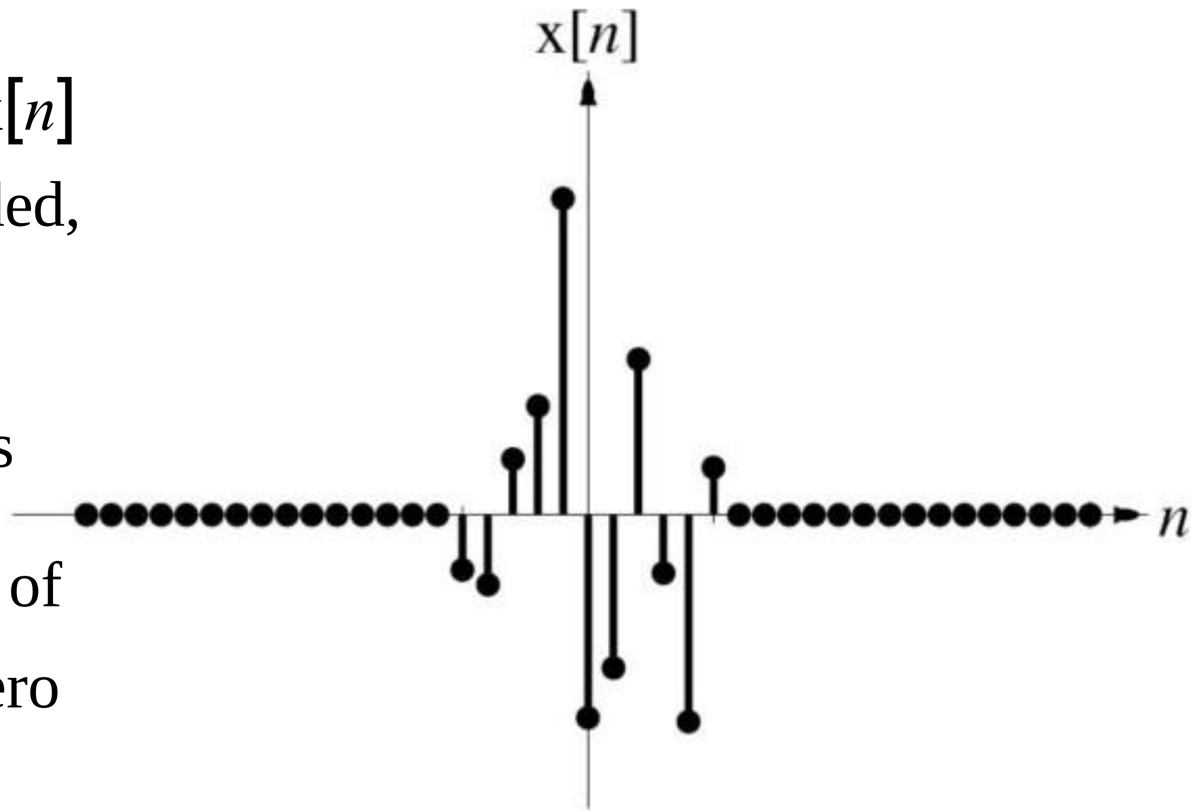
$$x[n] = \frac{1}{j2\pi} \oint_C X(z) z^{n-1} dz.$$

This is a contour integral in the complex plane and is beyond the scope of this course. The notation $x[n] \xleftrightarrow{z} X(z)$ indicates that $x[n]$ and $X(z)$ form a "z-transform pair".

Existence of the *z* Transform

Time Limited Signals

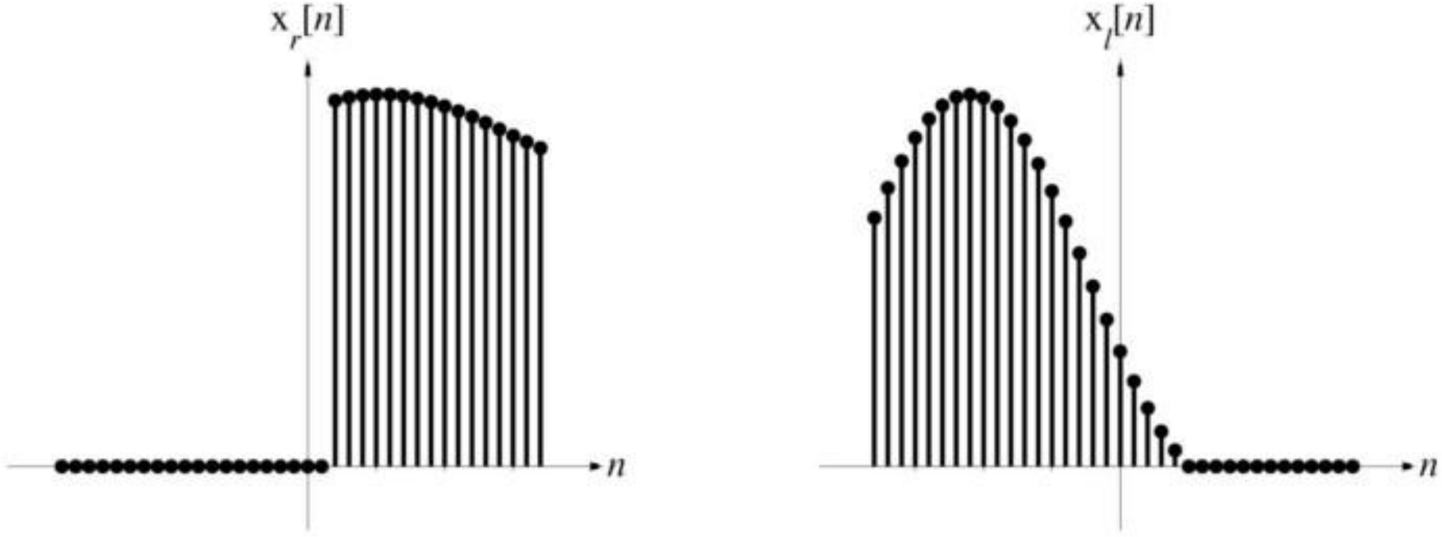
If a discrete-time signal x[n] is time limited and bounded, the z transformation summation $\sum_{n=-\infty}^{\infty} x[n] z^{-n}$ is finite and the z transform of x[n] exists for any non-zero value of z.



Existence of the z Transform

Right- and Left-Sided Signals

A right-sided signal $x_r[n]$ is one for which $x_r[n] = 0$ for any $n < n_0$ and a left-sided signal $x_l[n]$ is one for which $x_l[n] = 0$ for any $n > n_0$.

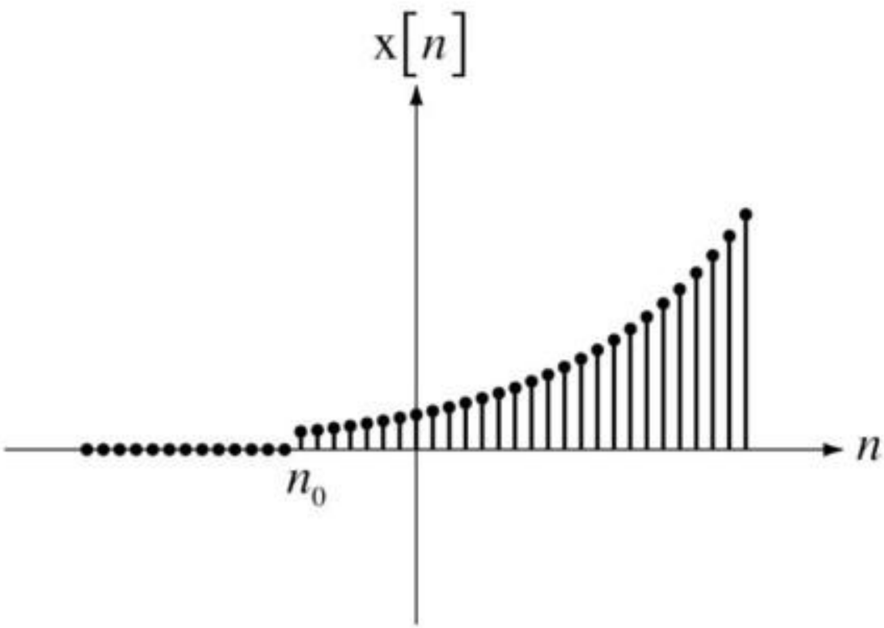


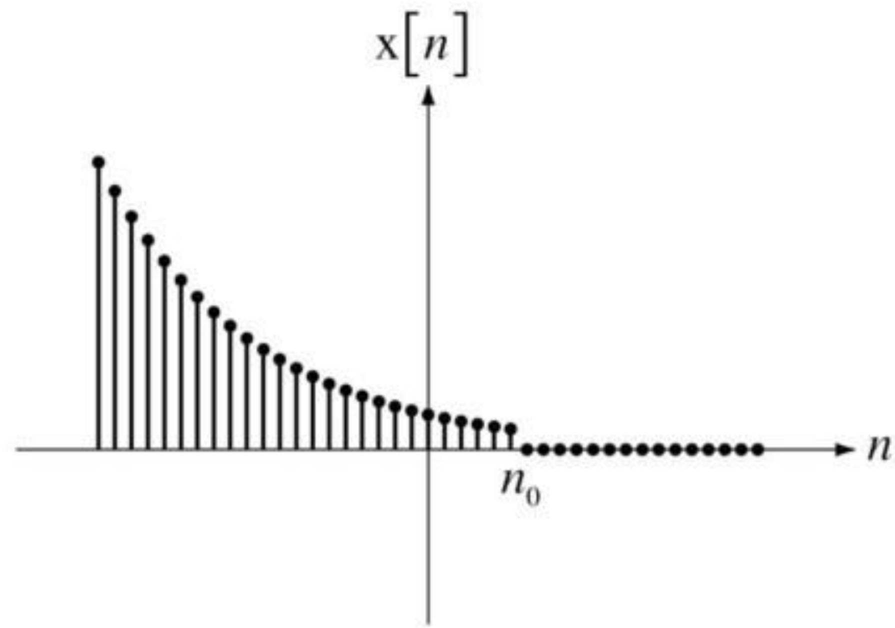
Existence of the *z* Transform

Right- and Left-Sided Exponentials

$x[n]=a_n\, u[n - n_0]$, a Î·

$x[n]= b_n\, u[n_0 - n]$, b Î·



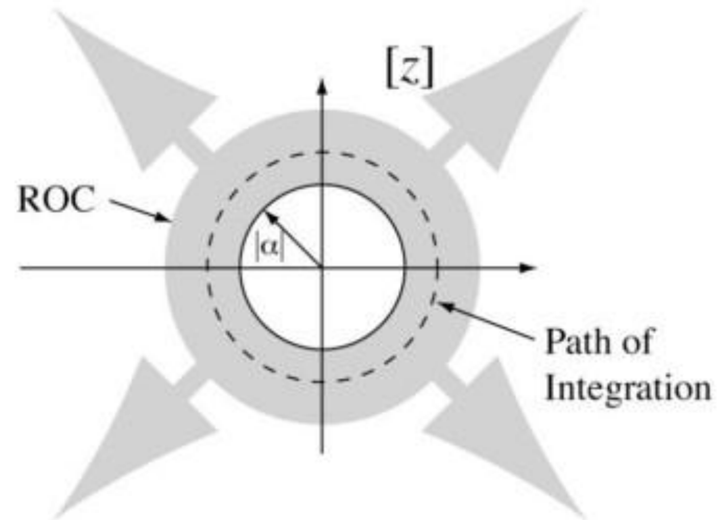


Existence of the *z* Transform

The *z* transform of $x[n] = \alpha^n u[n - n_0]$, $\alpha \in \mathbb{R}$ is

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n - n_0] z^{-n} = \sum_{n=n_0}^{\infty} \left(\alpha z^{-1}\right)^n$$

if the series converges and it converges if $|z| > |\alpha|$. The path of integration of the inverse *z* transform must lie in the region of the *z* plane outside a circle of radius $|\alpha|$

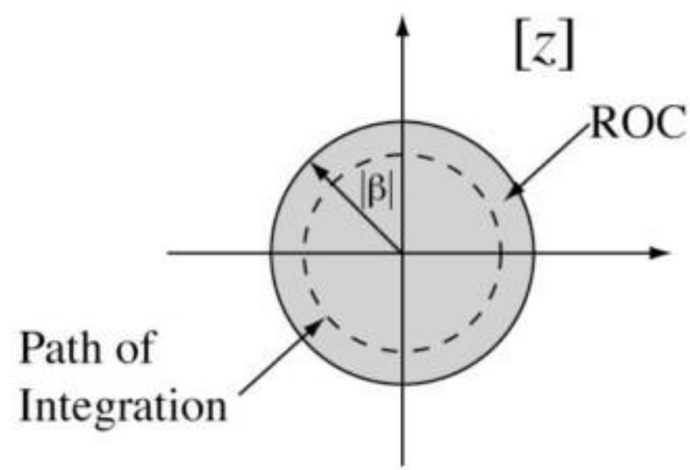


Existence of the z Transform

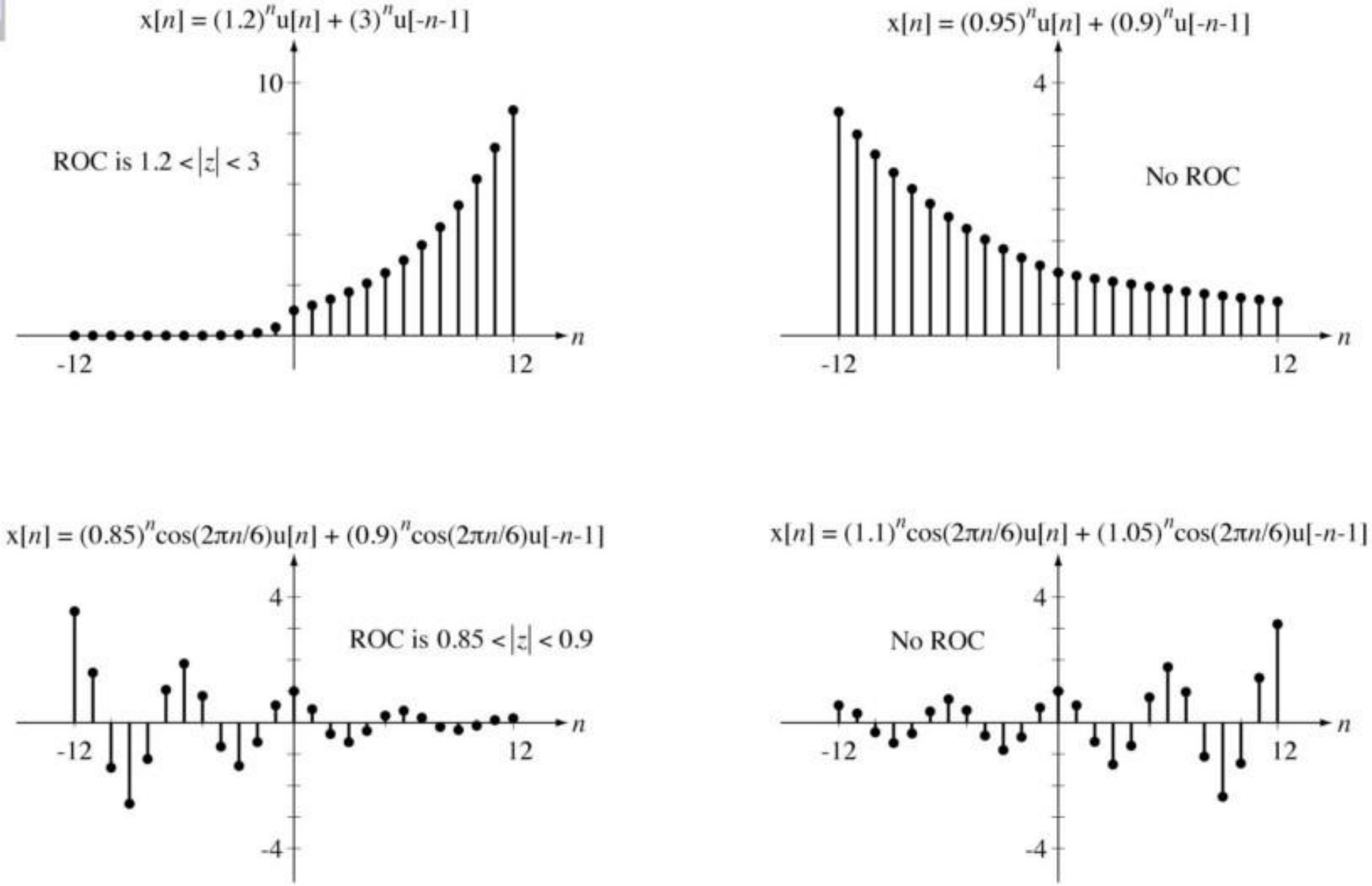
The z transform of $x[n] = \beta^n u[n_0 - n]$, $\beta \in \cdot$ is

$$X(z) = \sum_{n=-\infty}^{n_0} \beta^n z^{-n} = \sum_{n=-\infty}^{n_0} \left(\beta z^{-1}\right)^n = \sum_{n=-n_0}^{\infty} \left(\beta^{-1} z\right)^n$$

if the series converges and it converges if $|z| < |\beta|$. The path of integration of the inverse z transform must lie in the region of the z plane inside a circle of radius $|\beta|$



Existence of the z Transform



Some Common z Transform Pairs

$$\delta[n] \xleftrightarrow{\mathcal{Z}} 1 \quad , \text{All } z$$

$$\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1} = \frac{1}{1-z^{-1}} \; , \; |z|>1 \quad , \quad -\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1} \; , \; |z|<1$$

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \; , \; |z|>|\alpha| \quad , \quad -\alpha^n \mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \; , \; |z|<|\alpha|$$

$$n\,\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2} \; , \; |z|>1 \quad , \quad -n\,\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2} \; , \; |z|<1$$

$$n\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{\alpha z}{(z-\alpha)^2} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \; , \; |z|>|\alpha| \quad , \quad -n\alpha^n \mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{\alpha z}{(z-\alpha)^2} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \; , \; |z|<|\alpha|$$

$$\sin(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z\sin(\Omega_0)}{z^2-2z\cos(\Omega_0)+1} \; , \; |z|>1 \quad , \quad -\sin(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z\sin(\Omega_0)}{z^2-2z\cos(\Omega_0)+1} \; , \; |z|<1$$

$$\cos(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z[z-\cos(\Omega_0)]}{z^2-2z\cos(\Omega_0)+1} \; , \; |z|>1 \quad , \quad -\cos(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} = \frac{z[z-\cos(\Omega_0)]}{z^2-2z\cos(\Omega_0)+1} \; , \; |z|<1$$

$$\alpha^n \sin(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z\alpha\sin(\Omega_0)}{z^2-2\alpha z\cos(\Omega_0)+\alpha^2} \; , \; |z|>|\alpha| \quad , \quad -\alpha^n \sin(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z\alpha\sin(\Omega_0)}{z^2-2\alpha z\cos(\Omega_0)+\alpha^2} \; , \; |z|<|\alpha|$$

$$\alpha^n \cos(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z[z-\alpha\cos(\Omega_0)]}{z^2-2\alpha z\cos(\Omega_0)+\alpha^2} \; , \; |z|>|\alpha| \quad , \quad -\alpha^n \cos(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z[z-\alpha\cos(\Omega_0)]}{z^2-2\alpha z\cos(\Omega_0)+\alpha^2} \; , \; |z|<|\alpha|$$

$$\alpha^{|n|} \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} - \frac{z}{z-\alpha^{-1}} \; , \; |\alpha|<|z|<|\alpha^{-1}|$$

$$\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1}\left(z^{-n_0}-z^{-n_1}\right) = \frac{z^{n_1-n_0-1}+z^{n_1-n_0-2}+\cdots+z+1}{z^{n_1-1}} \; , \; |z|>0$$

z-Transform Properties

Given the z-transform pairs $g[n] \stackrel{Z}{\longleftrightarrow} G(z)$ and $h[n] \stackrel{Z}{\longleftrightarrow} H(z)$ with ROC's of ROC_G and ROC_H respectively the following properties apply to the z transform.

Linearity

$$\alpha g[n] + \beta h[n] \stackrel{Z}{\longleftrightarrow} \alpha G(z) + \beta H(z)$$

$$ROC = ROC_G \cap ROC_H$$

Time Shifting

$$g[n - n_0] \stackrel{Z}{\longleftrightarrow} z^{-n_0} G(z)$$

$$ROC = ROC_G \text{ except perhaps } z = 0 \text{ or } z \to \infty$$

Change of Scale in z

$$\alpha^n g[n] \stackrel{Z}{\longleftrightarrow} G(z/\alpha)$$

$$ROC = |\alpha| ROC_G$$

z-Transform Properties

Time Reversal

$$g[-n] \xleftrightarrow{\ \mathcal{Z}\ } G(z^{-1})$$

$$\text{ROC} = 1/\text{ROC}_G$$

Time Expansion

$$\begin{Bmatrix} g[n/k] & , \; n/k \text{ and integer} \\ 0 & , \; \text{otherwise} \end{Bmatrix} \xleftrightarrow{\ \mathcal{Z}\ } G(z^k)$$

$$\text{ROC} = (\text{ROC}_G)^{1/k}$$

Conjugation

$$g^*[n] \xleftrightarrow{\ \mathcal{Z}\ } G^*(z^*)$$

$$\text{ROC} = \text{ROC}_G$$

***z*-Domain Differentiation**

$$-n\,g[n] \xleftrightarrow{\ \mathcal{Z}\ } z\frac{d}{dz}G(z)$$

$$\text{ROC} = \text{ROC}_G$$

z-Transform Properties

Convolution $\quad g[n]*h[n] \xleftrightarrow{Z} H(z)G(z)$

First Backward Difference $\quad g[n]-g[n-1] \xleftrightarrow{Z} (1-z^{-1})G(z)$

$$\text{ROC} \supseteq \text{ROC}_G \cap |z|>0$$

Accumulation $\quad \sum_{m=-\infty}^{n} g[m] \xleftrightarrow{Z} \frac{z}{z-1}G(z)$

$$\text{ROC} \supseteq \text{ROC}_G \cap |z|>1$$

Initial Value Theorem $\quad$ If $g[n]=0\ ,\ n<0$ then $g[0]=\lim_{z\to\infty} G(z)$

Final Value Theorem $\quad$ If $g[n]=0\ ,\ n<0,\ \lim_{n\to\infty} g[n]=\lim_{z\to 1}(z-1)G(z)$

if $\lim_{n\to\infty} g[n]$ exists.

z-Transform Properties

For the final-value theorem to apply to a function $G(z)$ all the finite poles of the function $(z - 1)G(z)$ must lie in the open interior of the unit circle of the z plane. Notice this does not say that all the poles of $G(z)$ must lie in the open interior of the unit circle. $G(z)$ could have a single pole at $z = 1$ and the final-value theorem could still apply.

The Inverse *z* Transform

Synthetic Division

For rational *z* transforms of the form

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

we can always find the inverse *z* transform by synthetic division. For example,

$$H(z) = \frac{(z-1.2)(z+0.7)(z+0.4)}{(z-0.2)(z-0.8)(z+0.5)} \quad , \quad |z| > 0.8$$

$$H(z) = \frac{z^3 - 0.1z^2 - 1.04z - 0.336}{z^3 - 0.5z^2 - 0.34z + 0.08} \quad , \quad |z| > 0.8$$

The Inverse *z* Transform

Synthetic Division

$$
\begin{array}{r}
1+0.4z^{-1}+0.5z^{-2}\cdots \\
z^3-0.5z^2-0.34z+0.08\,\overline{)\,z^3-0.1z^2-1.04z-0.336} \\
\underline{z^3-0.5z^2-0.34z+0.08} \\
0.4z^2-\ 0.7z-\ 0.256 \\
\underline{0.4z^2\ -0.2z-\ \ 0.136-0.032z^{-1}} \\
0.5z-\ \ 0.12+0.032z^{-1} \\
\vdots \qquad \vdots \qquad \vdots
\end{array}
$$

The inverse *z* transform is

$$\delta[n]+0.4\delta[n-1]+0.5\delta[n-2]\cdots \xleftrightarrow{\mathcal{Z}} 1+0.4z^{-1}+0.5z^{-2}\cdots$$

The Inverse z Transform

Synthetic Division

We could have done the synthetic division this way.

$$
\begin{array}{r}
-4.2 - 30.85z - 158.613z^2 \cdots \\
0.08 - 0.34z - 0.5z^2 + z^3 \overline{\big) -0.336 - 1.04z - 0.1z^2 + z^3} \\
\underline{-0.336 + 1.428z + 2.1z^2 - 4.2z^3} \\
-2.468z - 2.2z^2 + 5.2z^3 \\
\underline{-2.468z + 10.489z^2 + 15.425z^3 - 30.85z^4} \\
-12.689z^2 - 10.225z^3 + 30.85z^4 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}
$$

$$
-4.2\delta[n] - 30.85\delta[n+1] - 158.613\delta[n+2] \cdots \overset{z}{\longleftrightarrow} -4.2 - 30.85z - 158.613z^2 \cdots
$$

but with the restriction $|z| > 0.8$ this second form does not converge and is therefore not the inverse z transform.

The Inverse *z* Transform

Synthetic Division

We can always find the inverse *z* transform of a rational function with synthetic division but the result is not in closed form. In most practical cases a closed-form solution is preferred.

Partial Fraction Expansion

Partial-fraction expansion works for inverse z transforms the same way it does for inverse Laplace transforms. But there is a situation that is quite common in inverse z transforms which deserves mention. It is very common to have z-domain functions in which the number of finite zeros equals the number of finite poles (making the expression improper in z) with at least one zero at $z = 0$.

$$H(z) = \frac{z^{N-M}(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

Partial Fraction Expansion

Dividing both sides by z we get

$$\frac{\mathrm{H}(z)}{z}=\frac{z^{N-M-1}(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

and the fraction on the right is now proper in z and can be expanded in partial fractions.

$$\frac{\mathrm{H}(z)}{z}=\frac{K_1}{z-p_1}+\frac{K_2}{z-p_2}+\cdots+\frac{K_N}{z-p_N}$$

Then both sides can be multiplied by z and the inverse transform can be found.

$$\mathrm{H}(z)=\frac{zK_1}{z-p_1}+\frac{zK_2}{z-p_2}+\cdots+\frac{zK_N}{z-p_N}$$

$$\mathrm{h}[n]=K_1 p_1^n\,\mathrm{u}[n]+K_2 p_2^n\,\mathrm{u}[n]+\cdots+K_N p_N^n\,\mathrm{u}[n]$$

z-Transform Properties

An LTI system has a transfer function

$$H(z) = \frac{Y(z)}{X(z)} \quad \frac{-1/2}{z^2 z - z + 2/9}, \quad |z| > 2/3$$

Using the time-shifting property of the z transform draw a block diagram realization of the system.

$$Y(z)(z^2 - z + 2/9) = X(z)(z - 1/2)$$

$$z^2 Y(z) = z X(z) - (1/2) X(z) + z Y(z) - (2/9) Y(z)$$

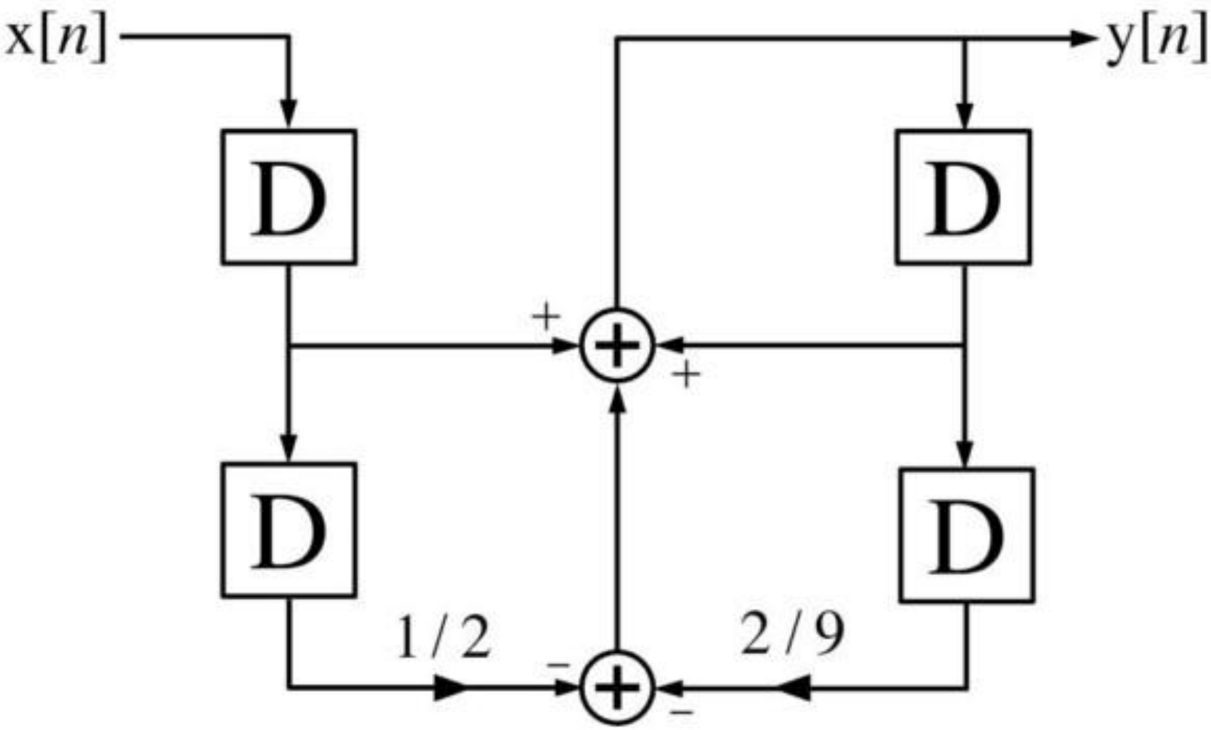
$$Y(z) = z^{-1} X(z) - (1/2) z^{-2} X(z) + z^{-1} Y(z) - (2/9) z^{-2} Y(z)$$

z-Transform Properties

$$Y(z) = z^{-1}X(z) - (1/2)z^{-2}X(z) + z^{-1}Y(z) - (2/9)z^{-2}Y(z)$$

Using the time-shifting property

$$y[n] = x[n-1] - (1/2)x[n-2] + y[n-1] - (2/9)y[n-2]$$



z-Transform Properties

Let $g[n] \xleftrightarrow{\mathcal{Z}} G(z) = \dfrac{z-1}{\left(z-0.8e^{-j\pi/4}\right)\left(z-0.8e^{+j\pi/4}\right)}$. Draw a pole-zero diagram for $G(z)$ and for the z transform of $e^{j\pi n/8} g[n]$.

The poles of $G(z)$ are at $z = 0.8e^{\pm j\pi/4}$ and its single finite zero is at $z = 1$. Using the change of scale property

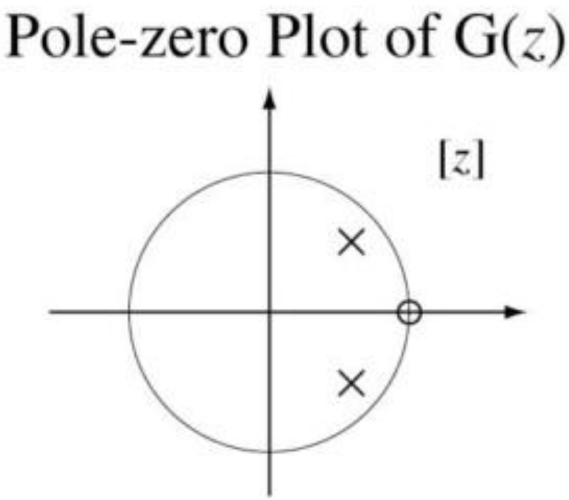
$$e^{j\pi n/8} g[n] \xleftrightarrow{\mathcal{Z}} G\left(ze^{-j\pi/8}\right) = \frac{ze^{-j\pi/8} - 1}{\left(ze^{-j\pi/8} - 0.8e^{-j\pi/4}\right)\left(ze^{-j\pi/8} - 0.8e^{+j\pi/4}\right)}$$

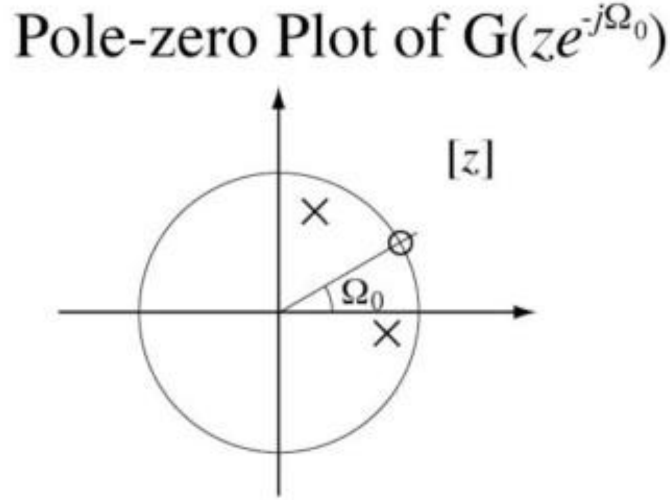
$$G\left(ze^{-j\pi/8}\right) = \frac{e^{-j\pi/8}\left(z - e^{j\pi/8}\right)}{e^{-j\pi/8}\left(z-0.8e^{-j\pi/8}\right)e^{-j\pi/8}\left(z-0.8e^{+j3\pi/8}\right)}$$

$$G\left(ze^{-j\pi/8}\right) = e^{j\pi/8}\frac{z - e^{j\pi/8}}{\left(z-0.8e^{-j\pi/8}\right)\left(z-0.8e^{+j3\pi/8}\right)}$$

z-Transform Properties

$G\left(ze^{-j\pi/8}\right)$ has poles at $z = 0.8e^{-j\pi/8}$ and $0.8e^{+j3\pi/8}$ and a zero at $z = e^{j\pi/8}$. All the finite zero and pole locations have been rotated in the z plane by $\pi/8$ radians.





z-Transform Properties

Using the accumulation property and $\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \dfrac{z}{z-1} \quad , \quad |z|>1$

show that the z transform of $n\,\mathrm{u}[n]$ is $\dfrac{z}{(z-1)^2} \quad , \quad |z|>1.$

$$n\,\mathrm{u}[n] = \sum_{m=0}^{n} \mathrm{u}[m-1]$$

$$\mathrm{u}[n-1] \xleftrightarrow{\mathcal{Z}} z^{-1}\frac{z}{z-1} = \frac{1}{z-1} \quad , \quad |z|>1$$

$$n\,\mathrm{u}[n] = \sum_{m=0}^{n} \mathrm{u}[m-1] \xleftrightarrow{\mathcal{Z}} \left(\frac{z}{z-1}\right)\frac{1}{z-1} = \frac{z}{(z-1)^2} \quad , \quad |z|>1$$

Inverse z Transform Example

Find the inverse z transform of

$$X(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad 0.5 < |z| < 2$$

Right-sided signals have ROC's that are outside a circle and left-sided signals have ROC's that are inside a circle. Using

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| > |\alpha|$$

$$-\alpha^n \mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| < |\alpha|$$

We get

$$(0.5)^n \mathrm{u}[n] + (-2)^n \mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} X(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad 0.5 < |z| < 2$$

Inverse z Transform Example

Find the inverse z transform of

$$\mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| > 2$$

In this case, both signals are right sided. Then using

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| > |\alpha|$$

We get

$$\left[(0.5)^n - (-2)^n\right]\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| > 2$$

Inverse z Transform Example

Find the inverse z transform of

$$\mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| < 0.5$$

In this case, both signals are left sided. Then using

$$-\alpha^n \,\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| < |\alpha|$$

We get

$$-\left[(0.5)^n - (-2)^n\right]\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| < 0.5$$

The Unilateral z Transform

Just as it was convenient to define a unilateral Laplace transform it is convenient for analogous reasons to define a unilateral z transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Properties of the Unilateral *z* Transform

If two causal discrete-time signals form these transform pairs,

$\mathrm{g}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{G}(z)$ and $\mathrm{h}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{H}(z)$ then the following properties hold for the unilateral z transform.

Time Shifting

Delay: $\mathrm{g}[n-n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0}\,\mathrm{G}(z)\,,\ n_0 \geq 0$

Advance: $\mathrm{g}[n+n_0] \xleftrightarrow{\mathcal{Z}} z^{n_0}\left(\mathrm{G}(z) - \sum_{m=0}^{n_0-1} \mathrm{g}[m] z^{-m}\right)\,,\ n_0 > 0$

Accumulation:

$$\sum_{m=0}^{n} \mathrm{g}[m] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1}\mathrm{G}(z)$$

Solving Difference Equations

The unilateral z transform is well suited to solving difference equations with initial conditions. For example,

$$y[n+2] - \frac{3}{2} y[n+1] + \frac{1}{2} y[n] = (1/4)^n \quad , \text{ for } n \geq 0$$

$$y[0] = 10 \quad \text{and} \quad y[1] = 4$$

z transforming both sides,

$$z^2 \left[Y(z) - y[0] - z^{-1} y[1] \right] - \frac{3}{2} z \left[\quad \right] + \frac{1}{2} Y(z) = \frac{z}{z - 1/4}$$

the initial conditions are called for systematically.

Solving Difference Equations

Applying initial conditions and solving,

$$Y(z) = z\left(\frac{16/3}{z-1/4} + \frac{4}{z-1/2} + \frac{2/3}{z-1}\right)$$

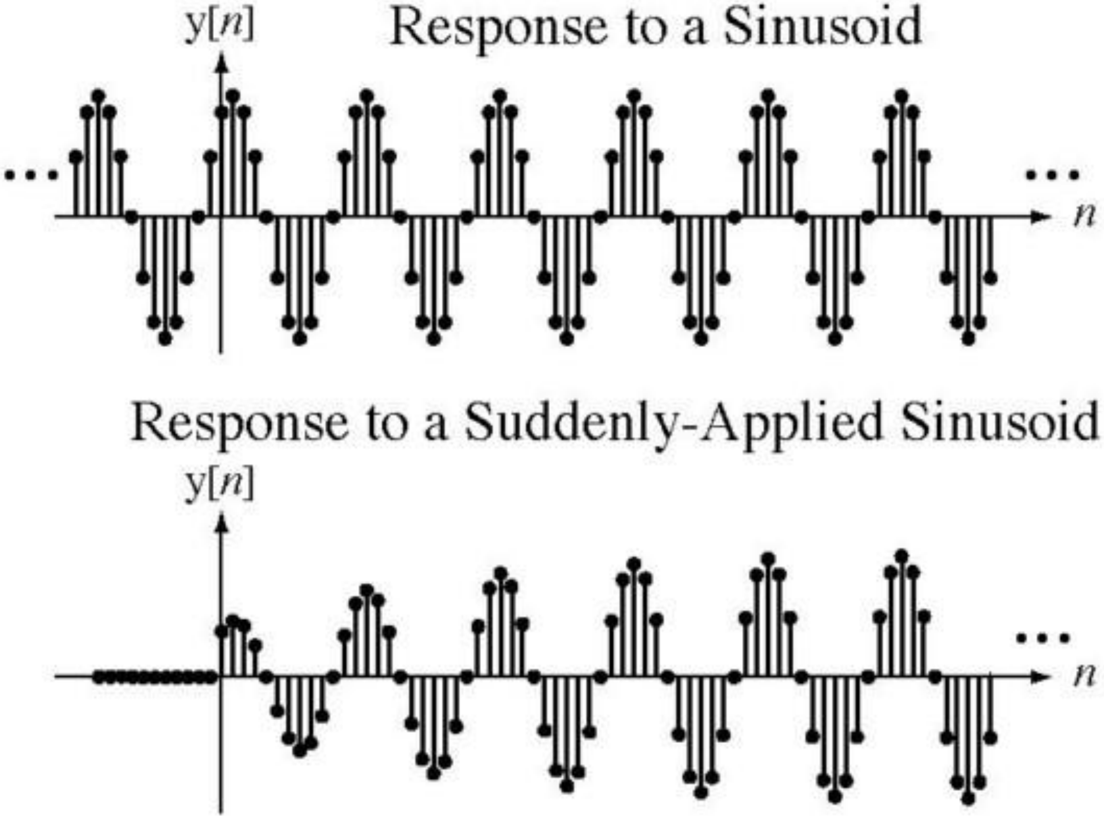
and

$$y[n] = \left[\frac{16}{3}\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{2}\right)^n + \frac{2}{3}\right]u[n]$$

This solution satisfies the difference equation and the initial conditions.

Pole-Zero Diagrams and Frequency Response

For a stable system, the response to a sinusoid applied at time *t* = 0 approaches the response to a true sinusoid (applied for all time).



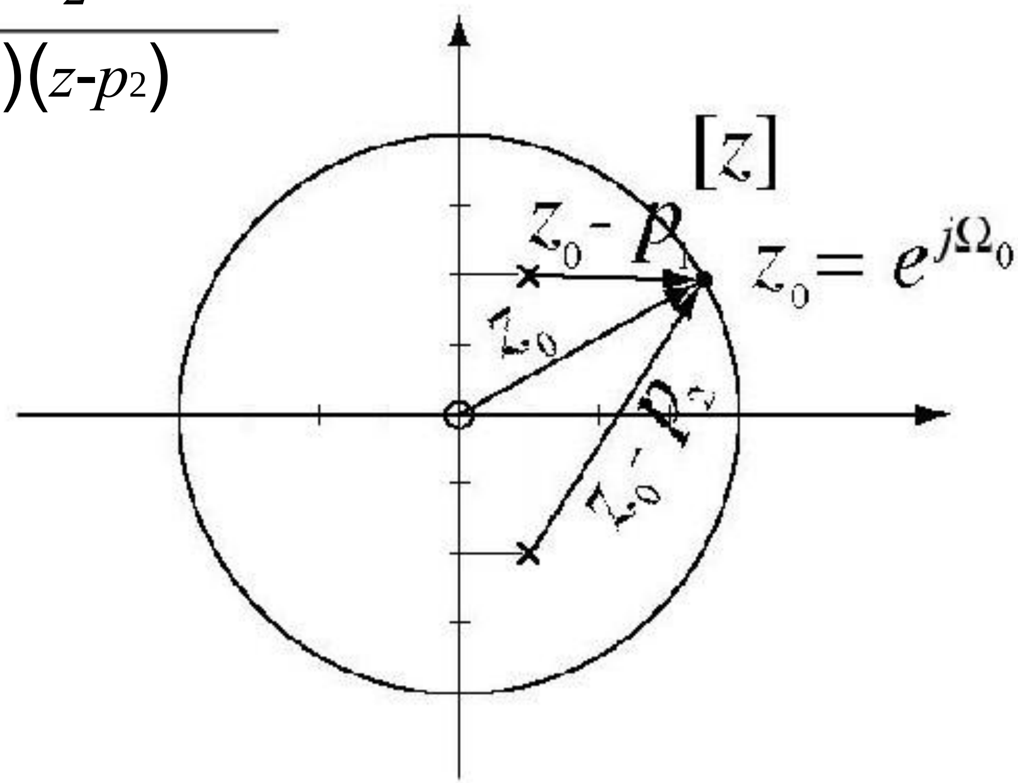
Pole-Zero Diagrams and Frequency Response

Let the transfer function of a system be

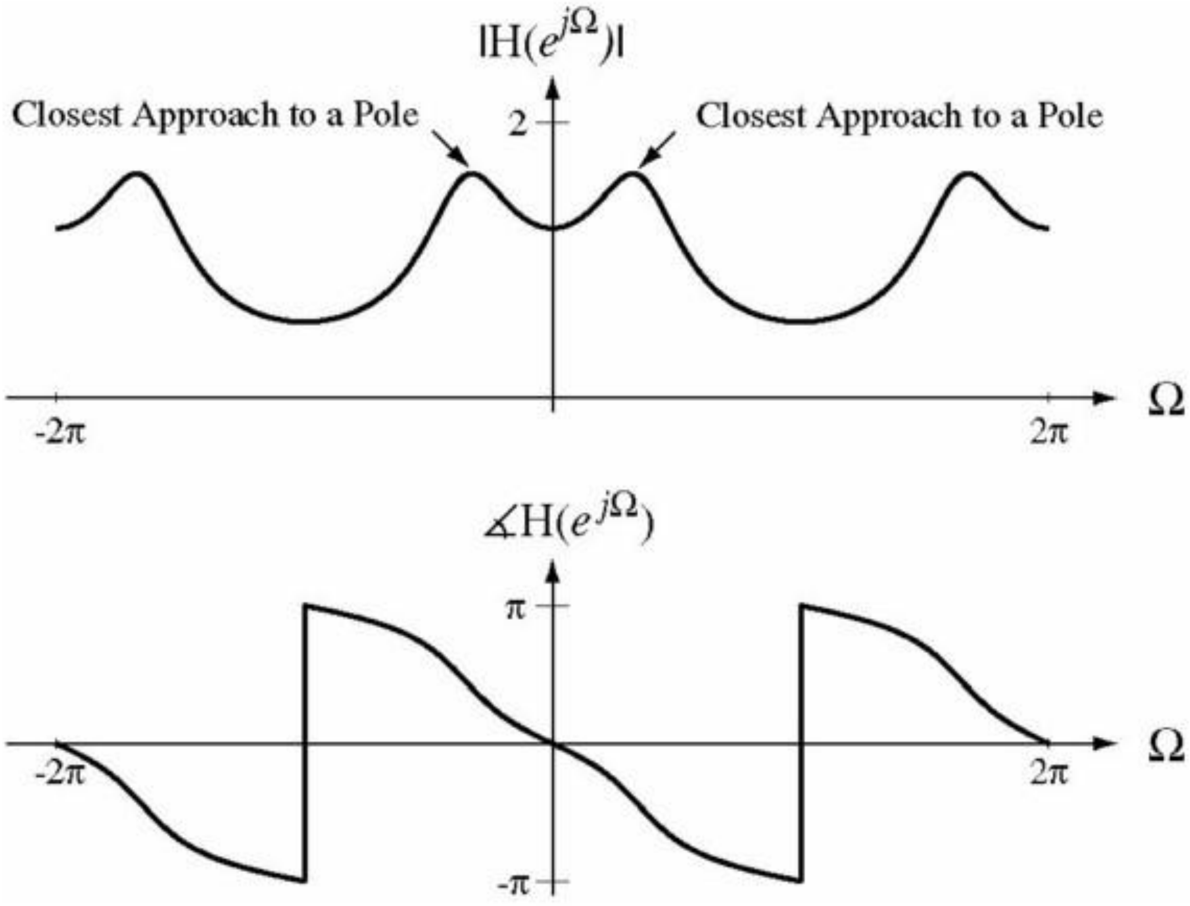
$$H(z) = \frac{z}{z^2 - z/2 + 5/16} = \frac{z}{(z-p_1)(z-p_2)}$$

$$p_1 = \frac{1+j2}{4}, \quad p_2 = \frac{1-j2}{4}$$

$$\left|H(e^{jW})\right| = \frac{\left|e^{jW}\right|}{\left|e^{jW} - p_1\right| \left|e^{jW} - p_2\right|}$$



Pole-Zero Diagrams and Frequency Response



Transform Method Comparison

A system with transfer function $H(z) = \dfrac{z}{(z-0.3)(z+0.8)}, \quad |z| > 0.8$

is excited by a unit sequence. Find the total response.

Using z-transform methods,

$$Y(z) = H(z)X(z) = \frac{z}{(z-0.3)(z+0.8)} \times \frac{\;}{z-1}, \quad |z| > 1$$

$$Y(z) = \frac{z^2}{\;} \quad \frac{.1169}{z-0.3} + \frac{0.3232}{z+0.8} + \frac{0.7937}{z-1}, \quad |z| > 1$$

$$y[n] = \left[-0.1169\,(0.3)^{n-1} + 0.3232\,(-0.8)^{n-1} + 0.7937\right] u[n-1]$$

Transform Method Comparison

Using the DTFT,

$$\mathrm{H}\left(e^{j\Omega}\right)=\frac{e^{j\Omega}}{\left(e^{j\Omega}-0.3\right)\left(e^{j\Omega}+0.8\right)}$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\mathrm{H}\left(e^{j\Omega}\right)\mathrm{X}\left(e^{j\Omega}\right)=\frac{e^{j\Omega}}{\left(e^{j\Omega}-0.3\right)\left(e^{j\Omega}+0.8\right)}\times\underbrace{\left(\frac{1}{1-e^{-j\Omega}}+\pi\delta_{2\pi}(\Omega)\right)}_{\text{DTFT of a Unit Sequence}}$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{e^{j2\Omega}}{\left(e^{j\Omega}-0.3\right)\left(e^{j\Omega}+0.8\right)\left(e^{j\Omega}-1\right)}+\pi\frac{e^{j\Omega}}{\left(e^{j\Omega}-0.3\right)\left(e^{j\Omega}+0.8\right)}\delta_{2\pi}(\Omega)$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{-0.1169}{e^{j\Omega}-0.3}+\frac{0.3232}{e^{j\Omega}+0.8}+\frac{0.7937}{e^{j\Omega}-1}+\frac{\pi}{(1-0.3)(1+0.8)}\delta_{2\pi}(\Omega)$$

Transform Method Comparison

Using the equivalence property of the impulse and the periodicity of both $\delta_{2\pi}(\Omega)$ and $e^{j\Omega}$

$$Y\left(e^{j\Omega}\right)=\frac{-0.1169e^{-j\Omega}}{1-0.3e^{-j\Omega}}+\frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}}+\frac{0.7937e^{-j\Omega}}{1-e^{-j\Omega}}+2.4933\delta_{2\pi}(\Omega)$$

Then, manipulating this expression into a form for which the inverse DTFT is direct

$$Y\left(e^{j\Omega}\right)=\frac{-0.1169e^{-j\Omega}}{1-0.3e^{-j\Omega}}+\frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}}+0.7937\left(\frac{e^{-j\Omega}}{1-e^{-j\Omega}}+\pi\delta_{2\pi}(\Omega)\right)$$

$$\underbrace{-0.7937\pi\delta_{2\pi}(\Omega)+2.4933\delta_{2\pi}(\Omega)}_{=0}$$

Transform Method Comparison

$$Y\left(e^{j\Omega}\right) = \frac{-0.1169e^{-j\Omega}}{1-0.3e^{-j\Omega}} + \frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}} + 0.7937\left(\frac{e^{-j\Omega}}{1-e^{-j\Omega}} + \pi\delta_{2\pi}(\Omega)\right)$$

Finding the inverse DTFT,

$$y[n] = \left[-0.1169(0.3)^{n-1} + 0.3232(-0.8)^{n-1} + 0.7937\right]u[n-1]$$

The result is the same as the result using the *z* transform, but the effort and the probability of error are considerably greater.

System Response to a Sinusoid

A s tem with transfer function

$$H(z) = \frac{z}{z - 0.9} \quad , \quad |z| > 0.9$$

is excited by the sinusoid $x[n] = \cos(2\pi n / 12)$. Find the response.

The z transform of a true sinusoid does not appear in the table of z transforms. The z transform of a causal sinusoid of the form $x[n] = \cos(2\pi n / 12)u[n]$ does appear. We can use the DTFT to find the response to the true sinusoid and the result is

$y[n] = 1.995\cos(2\pi n / 12 - 1.115)$.

System Response to a Sinusoid

Using the z transform we can find the response of the system to a causal sinusoid $x[n] = \cos(2p\, n / 12)u[n]$ and the response is

$y[n]= 0.1217(0.9)^n u[n]+1.995\cos(2pn /12 -1.115)u[n]$

Notice that the response consists of two parts, a transient response $0.1217(0.9)^n u[n]$ and a forced response $1.995 \cos(2p\, n / 12 - 1.115)u[n]$ that, except for the unit sequence factor, is exactly the same as the forced response we found using the DTFT.

System Response to a Sinusoid

This type of analysis is very common. We can generalize it to say that if a system has a transfer function $H(z) = \frac{N(z)}{D(z)}$ that the response to a causal cosine excitation $\cos(\Omega_0 n)u[n]$ is

$$y[n] = \underbrace{\mathcal{Z}^{-1}\left(z \frac{N_1(z)}{D(z)} \right)}_{\text{Natural or Transient Response}} + \underbrace{|H(p_1)| \cos(\Omega_0 n + \measuredangle H(p_1)) u[n]}_{\text{Forced Response}}$$

where $p_1 = e^{j\Omega_0}$. This consists of a natural or transient response and a forced response. If the system is stable the transient response dies away with time leaving only the forced response which, except for the $u[n]$ factor is the same as the forced response to a true cosine. So we can use the z transform to find the response to a true sinusoid.